Precision of distance determination using 3D to 2D projections:
The error of migration measurement using X-ray images

K. Burckhardt\textsuperscript{a}, Ch. Gerber\textsuperscript{b}, J. Hodler\textsuperscript{b}, H. Nötzli\textsuperscript{b}, G. Székely\textsuperscript{a,*}

\textsuperscript{a}Computer Vision Group, Communication Technology Lab, Swiss Federal Institute of Technology, Zürich, Switzerland
\textsuperscript{b}University Hospital Balgrist, Zürich, Switzerland

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Abstract

The goal of this study has been to objectively and reliably estimate the precision of measuring 2D migration of hip prostheses. This is the distance change over time between the implant and the bone observable in X-ray images. To reach this goal, a generally valid scheme for determining the standard deviation of distance measurements in 3D to 2D projections has been worked out. The scheme was applied to four previously published methods for measuring the migration of the prosthetic cup using standard radiographs. Applying the scheme yields measures for the sensitivity of the migration measurement towards the relevant sources of error. Inserting previously published data for the amounts of the entering errors, the standard deviation of the migration measurement has been calculated numerically resulting in values up to several millimeters. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Total hip replacement is the most successful orthopaedic operation. In Europe alone, about 400,000 artificial hip joints are implanted per year. However, the problem of the fixation of the implants in the bone is not yet solved. After implantation, standard radiographs of the hip are routinely performed for the confirmation or exclusion of implant loosening. A number of well-known radiological signs have been described for this purpose: radiolucent zones at the cement-bone or implant-bone interface, fracture of cement, and migration of the implant (Harris et al., 1982; Bruijn et al., 1995; Seelen et al., 1995; Manaster, 1996). These signs become detectable with the naked eye in advanced stages of loosening only, maybe several years after surgery, and their interpretation is highly observer dependent. Therefore, if a certain type of implant has an abnormal tendency to loosen, a large number of prostheses may have been inserted before the implant’s problem becomes apparent.

Measured migration – the observed increase or decrease of the distance between the implant and the bone – is “generally accepted as a radiographic sign for prosthetic loosening” (Ilchmann et al., 1998b, p. 380). Thus, quantification of implant migration provides an alternative for the assessment of loosening and for a quantitative judgment of the implant’s design and the implantation technique. In (Sutherland et al., 1982), the prosthetic cup is considered loose if the 2D migration, which is the change of the bone-implant distance projected on the radiographic film, has exceeded 5 mm. Especially early postoperative migration is supposed to give information about the (later) fixation of the implant in the bone (Krismer et al., 1996; Nilsson and Kärrholm, 1996; Kärrholm et al., 1997). Nilsson and Kärrholm (1996) recapitulated that the risk of a failed ingrowth is minimal if the migration is less than 1 mm and occurs only in the first year after surgery. Measuring 2D migration of two different cups, Krismer et al. (1996) found that a total migration in the first 2 years greater than 1 mm predicts a high risk of loosening.
Investigating the coherence between loosening and 2D migration the error of the migration measurement needs to be known. There are many attempts to experimentally estimate the precision of measuring 2D migration (Nunn et al., 1989; Wetherell et al., 1989; Sutherland and Bresina, 1992; Ilchmann et al., 1992a,b, 1998a; Malchau et al., 1995), which have led to contradictory statements: For example, Nunn et al. (1989) estimated the error of their own method to be 3.0 mm, whereas for the same method Ilchmann et al. (1992b) stated a standard deviation of 0.7 mm. Reliable estimation of the overall measurement error would need an unrealistic amount of experimental data. Instead, one should aim at tracing it back to the individual error components, where realistic error bound estimates can be gained more easily.

The following well-known sources of error play a role in measuring 2D migration using radiographs: the variability of the extrinsic parameters (here the position and orientation of the patient’s pelvis at exposure), the unknown intrinsic parameters (here the film–focus distance and the intersection point of the vertical beam with the film plane), and the lack of definite reference points in the image (here an implant reference point and bony landmarks).

In this study, the overall error of 2D migration measurement was determined theoretically, taking into account the individual error components which arise from the above-mentioned sources of error. In doing so a general scheme was obtained for reliably and objectively estimating the a priori standard deviation of distance measurements in X-ray images. The scheme additionally yields information about the sensitivity of the measured migration towards the sources of error. As it is based on a statistical approach, only knowledge about the random error of the migration measurement could be gained whereas the bias was disregarded.

We applied the scheme to methods for measuring cup migration which previously were investigated experimentally: to the methods proposed by Nunn et al. (1989), by Dickob et al. (1994), by Sutherland et al. (1982) and to EBRA (Ein-Bild-Röntgen-Analyse), a method proposed by Krismer et al. (1995).

2. The analysed methods for measuring 2D migration

In 2D migration measurement methods the X-ray images of the hip replacement’s follow-up study are evaluated. The changes of the x- and y-distance, \(d_x\) and \(d_y\), between the implant and certain x- and y-reference-points, -lines or -planes defined by bony structures are determined. The \(x\)-distance corresponds to the medio-lateral, the \(y\)-distance to the crano-caudal distance. The landmarks of the implant and of the surrounding bone are usually marked in the image by hand using a pencil in case of conventional films and by mouse-clicking if digitized radiographs are used.

The methods analysed in this study can be classified as simple (Sutherland et al., 1982; Nunn et al., 1989; Dickob et al., 1994) and as complex (Krismer et al., 1995). The simple methods are based on the measurement of image distances and subsequent corrections for magnification and eventually also for rotation. In (Krismer et al., 1995) the distances between the 3D position of the implant and two planes, defined by bony landmarks and assumed intrinsic parameters, are determined.

In the following, the methods analysed in this study are shortly summarized. The notation shown in the figures is explained in the text.

2.1. 2D migration according to Nunn et al.

In (Nunn et al., 1989), the left and the right teardrop figure, which are the projections of bony structures of the hip socket, are used as bony landmarks. They are in most of the cases still visible after implantation of a prosthesis. The recommended film–focus distance is 900 mm. A reference line is defined by connecting the most caudal points (\(m_1\) and \(m_2\)) of the teardrop figures (see Fig. 1). The implant reference point (\(c\)) is the center of the projection of the femoral head (with the assumption of a polyethylene cup). The measured \(x\)-distance (\(d_x\)) corresponds to the distance along the reference line between the most caudal point of the teardrop figure next to the prosthesis and the implant reference point. The measured \(y\)-distance (\(d_y\)) corresponds to the distance perpendicular to the reference line between the mentioned points.

The distances are corrected for magnification by multiplication by \(r/b\), where \(r\) is the true radius of the femoral head and \(b\) the radius of its projection measured in the radiograph. In addition to the \(x\)-y-distance, the distance between the most caudal points of the teardrop figures (in the following the inter-teardrop distance) is determined in all radiographs of the patient’s follow-up study. After correction for magnification, the \(x\)-distance is multiplied by the largest inter-teardrop distance and divided by the one in the radiograph under study to correct for rotation of the pelvis around the crano-caudal axis. For the \(y\)-distance, no correction for rotation is considered.

2.2. 2D migration according to Dickob et al.

Dickob et al. (1994) have proposed a very similar procedure to the one described above. The only difference is the use of digitized instead of conventional radiographs and of a different implant reference point. The implant reference point is chosen to be the center of the metal ring at the cup’s rim (see Fig. 2) in case of polyethylene cups. If a metal cup is implanted, the center of the cup’s equatorial plane is used instead.

2.3. 2D migration according to Sutherland et al.

In (Sutherland et al., 1982), the teardrop figure next to the prosthesis (\(m_1\)) and the Köhler line (the line in Fig. 2
connecting $m_2$ and $m_3$) are used as bony landmarks. The implant reference point ($c$) is the center of the metal ring at the cup’s rim. The measured $x$-distance ($d_x$) corresponds to the distance between the implant reference point and the Köhler line. The measured $y$-distance ($d_y$) – as far as it is described in (Sutherland et al., 1982) – just corresponds to the difference between the $y$-coordinates of the implant reference point and the teardrop figure’s most caudal point. The correction for magnification is the same as in (Nunn et al., 1989) and in (Dickob et al., 1994), whereas a correction for rotation is not considered.

**Fig. 1.** The $x$- and the $y$-distance according to Nunn et al. (1989) and Dickob et al. (1994). The implant reference point is chosen as described in (Nunn et al., 1989).

**Fig. 2.** The $x$- and the $y$-distance according to Sutherland et al. (1982). The foramina obturatoria are two symmetrical apertures in the pelvis surrounded by the pubic bone and the ischium. As in (Dickob et al., 1994), the implant reference point ($c$) is chosen to be the center of either the cup’s equatorial plane or the metal ring at the cup’s rim. Several points of the visible cup contour are marked and a least-squares estimation is applied to determine the exact position of the implant reference point in the image. The $x$-$y$-distance corresponds to the distance (averaged as described below) of the reconstructed 3D implant reference point ($C$) to an $x$- and to a $y$-reference-plane. In Fig. 3, the normals to the reference planes are shown ($n_x$, $n_y$).

The image coordinates of the bony landmarks and of the implant reference point are represented relative to the assumed intersection point of the vertical beam with the film. This point is chosen to be the midpoint of the line connecting the film center and the cranial end of the symphysis pubis in the image ($m_4$).

**2D migration according to EBRA**

In (Krismer et al., 1995), points on the contours of the foramina obturatoria ($m_1$, $m_2$, and $m_3$) are used as bony landmarks (Fig. 3). The foramina obturatoria are two symmetrical apertures in the pelvis surrounded by the pubic bone and the ischium. As in (Dickob et al., 1994), the implant reference point ($c$) is chosen to be the center of either the cup’s equatorial plane or the metal ring at the cup’s rim. Several points of the visible cup contour are marked and a least-squares estimation is applied to determine the exact position of the implant reference point in the image. The $x$-$y$-distance corresponds to the distance (averaged as described below) of the reconstructed 3D implant reference point ($C$) to an $x$- and to a $y$-reference-plane. In Fig. 3, the normals to the reference planes are shown ($n_x$, $n_y$).

The 3D implant reference point is reconstructed using its projection on the film, and the extremal values for the film–implant and for the film–focus distance at exposure described in the next paragraph. The same values for the film–focus distance, and a medio-lateral as well as a cranio-caudal tangent to the foramina obturatoria in the image are used to calculate the $x$- and the $y$-reference plane. The medio-lateral tangent is the line connecting the most caudal image points, i.e. $m_1$ and $m_2$, of the foramina obturatoria. The cranio-caudal tangent is the line which is perpendicular to the medio-lateral one and intersects the most medial image point of the foramen obturatum next to
the prosthesis. In Fig. 3, the cranio-caudal tangent is the line connecting $m_4$ and $m'_4$.

In order to get an idea of the spatial situation at exposure, the general extremal values of the distance between implant and film plane were determined experimentally (Russe, 1988; Krismer et al., 1995). Using these values, the known real radius of the femoral head, and the radius of the femoral head in the image, a maximal and a minimal value for the film–focus distance ($f^{\text{max}}$, $f^{\text{min}}$) are calculated for each radiograph. If the resulting maximal value exceeds 1300 mm or the minimal value falls short of 900 mm, $f^{\text{max}} = 1300$ mm and $f^{\text{min}} = 900$ mm is set.

The $x$- and the $y$-reference plane, the 3D implant reference point, and the distances between this point and the planes are calculated inserting once the minimal and once the maximal implant–film distance and the respective values for the film–focus distance. The final 2D distance is the mean of the resulting minimal and maximal $x$-$y$-distances.

After the determination of the 2D distance the following comparability algorithm is applied to reduce the influence of pelvis rotation between two exposures: a comparability limit is defined using additional image distances strongly affected by rotation between certain bony structures. The radiographs of the follow-up study are subdivided into pairs of comparable radiographs. These are radiographs between which the additional image distances do not differ more than a certain amount, the ‘comparability limit’, which is usually 2 or 3 mm. Thus, the rotation between the comparable radiographs is implicitly limited. The migration at a certain time is equated to the mean $x$-$y$-distance change in all pairs of comparable radiographs whose first exposure was made before and whose second exposure was made after the time point.

3. Scheme for the precision estimation of 2D migration

In this study, for each of the described methods a sum was derived consisting of components corresponding to the individual entering errors. As the procedure of deriving these expressions for the standard deviation of the measured $x$-$y$-distance in principle is the same for all methods, it has been generalized to a scheme.

The vector of the $x$-$y$-distance can be represented as a function $d = d(p_i, v_j) = (d_i(p_i, v_j), d_j(p_i, v_j))^T$ depending on parameters $p_i$ ($i = 1, \ldots, l$) and on random variables $v_j$ ($j = 1, \ldots, m$). The predetermined parameters mainly represent the pelvis anatomy and are supposed to remain constant while measuring migration. The random variables describe the sources of error. The expressions for the standard deviation are derived using the principle of error propagation. Inserting the variances of the random variables in the expressions yields the standard deviation of $d_i$ and of $d_j$ that has to be assumed, if the $x$-$y$-distance in the analysed methods is determined using different radiographs of the same pelvis (under the assumption that no migration has occurred).
Using the principle of error propagation the partial derivatives of $d$ with respect to the random variables need to be calculated. The partial derivatives are measures for the significance of the random variables for the standard deviation of the distance vector. The greater the value of the partial derivative with respect to a certain random variable, the greater the influence of the corresponding source of error on the precision of the measured $x$- and $y$-distance.

The present study is restricted to the analysis of the four described methods. However, the evolved scheme can equally be applied to other 2D migration measurements methods, e.g., (Collet et al., 1985) or (Wetherell et al., 1989), or more generally, to the determination of distances using a 2D image of a 3D object.

The scheme mainly consists of the following three steps: identification of the parameters and random variables, representation of the measured $x$-$y$-distance as a function, and application of the principle of error propagation to this function. It is presented in detail in this section, which is organized as follows. In Section 3.1 the parameters, the variables, and the used notation is introduced, in Section 3.2 the representation of each distance vector as function $d(p_i, v_j)$ is explained, and in Section 3.3 the principle of error propagation is briefly described.

3.1. The parameters and variables of the measured 2D distance

At first, the parameters $p_i$ of the function $d$ are described. The 3D coordinates of anatomical points are given in the ‘camera coordinate system’ at the assumed mean pelvis orientation and position in the X-ray unit (see Section 4.1). The ‘camera coordinate system’ is defined by the image coordinate system ($x$-$y$-axes) and the beam perpendicular to the image plane ($z$-axis). The origin of this system lies at the focus, i.e., the X-ray source (see Fig. 4). Altogether, the parameters are:

- the 3D coordinates $C_0$ of the reference point of the implant;
- the $n$ ($n = 2−4$) 3D coordinates $M_{10}, \ldots, M_{n0}$ of the bony structures (the landmarks defining the reference points, lines or planes are their projections);
- the 3D coordinates $Z$ of the center of rotation of the pelvis between two exposures;
- the known radius of the femoral head, $r$;
- the assumed mean (concerning (Krismer et al., 1995; Sutherland et al., 1982; Dickob et al., 1994)) or the given nominal (concerning (Nunn et al., 1989)) value $f$ of the film–focus distance;
- concerning EBRA, the predetermined extremal values $h_{\text{max}} = 270$ mm and $h_{\text{min}} = 180$ mm of the distance between 3D implant reference point and film, and the predefined limits 1300 mm and 900 mm for the estimated extremal values of the film–focus distance (Russe, 1988; Krismer et al., 1995).

The random variables $v_j$ of the function $d$ are the following:

- The variables describing the variability of the extrinsic parameters, i.e., the pelvis’ varying orientation and position at exposure relative to the above-mentioned mean orientation and position. The varying orientation

![Fig. 4. The projection of anatomical points on the film. Dotted lines indicate the variables representing the sources of error.](image-url)
is described by the angles \( \delta \alpha, \delta \beta \) and \( \delta \gamma \) of the pelvis’ rotation around the \( x-, y- \) and \( z- \)axis and the varying position by the translation vector \( \delta \mathbf{T} = (\delta T_x, \delta T_y, \delta T_z) \). The rotation angles and the translation vector are by definition zero at the mean orientation and position.

- The variables describing the error of the measurements in the X-ray image. These variables are represented by \( \delta c, \delta m_1, \ldots, \delta m_n \) and \( \delta \beta \), which correspond to the deviations from the expected value of, respectively, the measured image coordinates of the implant reference point, those of the bony landmarks, and the measured radius of the femoral head in the image.
- The variables describing the variability of the intrinsic parameters. These are the deviations \( \delta f \) and \( \delta \mathbf{T} = (\delta T_x, \delta T_y, \delta T_z) \) of the intrinsic parameters at exposure from their mean values. The mean values are given by the mean film–focus distance \( f \) and the mean intersection point of the perpendicular beam with the film, which is assumed to be the film center.

### 3.2. Representation of the measured x-y-distance as function

This subsection is based on the notation introduced in the previous subsection, and the description of the analysed methods in Section 2. The \( x-y \)-distance as determined in (Nunn et al., 1989; Dickob et al., 1994; Sutherland et al., 1982) can be expressed by relatively simple functions because these methods mainly consist of the measurement of image distances. On the other hand, the function representing the \( x-y \)-distance determined in (Krismer et al., 1995) is quite complicated.

The measured image coordinates of the implant reference point and the bony landmarks can be expressed for all four methods in the same way. The random 3D positions \( \mathbf{C} \) and \( \mathbf{M}_k \) of the pelvis points depend on their mean positions, the rotation, and the translation:

\[
\mathbf{C} = \mathbf{Z} + \mathbf{R} \cdot (\mathbf{C}_0 - \mathbf{Z}) + \delta \mathbf{T},
\]

\[
\mathbf{M}_k = \mathbf{Z} + \mathbf{R} \cdot (\mathbf{M}_{k0} - \mathbf{Z}) + \delta \mathbf{T}.
\]

The rotation is defined by the rotation matrix

\[
\mathbf{R} = \begin{pmatrix}
\cos(\delta \beta) \cos(\delta \gamma) & -\cos(\delta \beta) \sin(\delta \gamma) & \sin(\delta \beta) \\
\cos(\delta \alpha) \sin(\delta \gamma) + \cos(\delta \alpha) \cos(\delta \beta) \cos(\delta \gamma) & \cos(\delta \alpha) \cos(\delta \gamma) - \sin(\delta \alpha) \sin(\delta \beta) \cos(\delta \gamma) & -\sin(\delta \alpha) \sin(\delta \gamma) \\
\sin(\delta \alpha) \sin(\delta \beta) \cos(\delta \gamma) & \sin(\delta \alpha) \sin(\delta \beta) \sin(\delta \gamma) & \cos(\delta \alpha) \cos(\delta \beta)
\end{pmatrix}.
\]

The X-ray system corresponds to a camera, where the optical center or the focus is the X-ray source and the retinal or film plane is the plane of the radiographic film. In comparison to a standard camera, it has the peculiarity that the \( z \)-distance between object and film plane is constant, as the vertical distance between examination table and film cassette is fixed. This means that a variation of the film–focus distance \( \delta f \) leads to the same change of the \( z \)-distance of an anatomical point to the focus.

Generally, the projections of \( \mathbf{C} \) and \( \mathbf{M}_k \) on the film plane are given by

\[
f \cdot \left( \begin{array}{c}
\mathbf{C}_x \\
\mathbf{C}_y \\
\mathbf{C}_z
\end{array} \right)^T \quad \text{and} \quad f \cdot \left( \begin{array}{c}
\mathbf{M}_{kx} \\
\mathbf{M}_{ky} \\
\mathbf{M}_{kz}
\end{array} \right)^T.
\]

For Krismer et al. (1995); Sutherland et al. (1982); Dickob et al. (1994) \( f = 1100 \text{ mm} \) was assumed, for Nunn et al. (1989) the proposed nominal value \( f = 900 \text{ mm} \) was taken. Considering the variability of the intrinsic parameters and of locating the bony landmarks and the implant reference point in the image, the measured image coordinates used to determine the migration are (see Fig. 4)

\[
c = (f + \delta f) \cdot \left( \begin{array}{c}
\frac{\mathbf{C}_x}{\mathbf{C}_z + \delta \mathbf{C}_z} \\
\frac{\mathbf{C}_y}{\mathbf{C}_z + \delta \mathbf{C}_z} \\
\frac{\mathbf{C}_z}{\mathbf{C}_z + \delta \mathbf{C}_z}
\end{array} \right)^T + \delta \mathbf{c} + \delta \mathbf{T} \quad (1)
\]

and

\[
\mathbf{m}_k = (f + \delta f) \cdot \left( \begin{array}{c}
\frac{\mathbf{M}_{kx}}{\mathbf{M}_{kz} + \delta \mathbf{M}_{kz}} \\
\frac{\mathbf{M}_{ky}}{\mathbf{M}_{kz} + \delta \mathbf{M}_{kz}} \\
\frac{\mathbf{M}_{kz}}{\mathbf{M}_{kz} + \delta \mathbf{M}_{kz}}
\end{array} \right)^T + \delta \mathbf{m}_k + \delta \mathbf{T} \quad (2)
\]

In order to represent the measured distance as a function, the dependence of the femoral head’s contour in the image from \( \mathbf{C} \) is also required. The projection of the spherical femoral head is in fact an ellipse. The difference between the long and the short axis of this ellipse lies at about 0.2 mm (Russe, 1988). Here, the measured radius of the femoral head contour is equated to the ellipse’s short axis. The formula of the short axis was deduced by Russe (1988) and is

\[
b = \frac{f + \delta f}{\sqrt{\left( \frac{\mathbf{C}_z + \delta \mathbf{C}_z}{r} \right)^2 - 1}},
\]

where \( f \) is replaced by \( f + \delta f \), to take the variability of the film–focus distance at exposure into account. For \( r \) the real radius of the femoral head is inserted, which is assumed to be 16 mm.

#### 3.2.1. The function \( \mathbf{d} \) representing the distance vector in Nunn et al. and in Dickob et al.

The most caudal point of the left teardrop figure is indicated with \( \mathbf{m}_1 \) and the right one with \( \mathbf{m}_2 \). The correction for rotation of the measured \( x \)-component corresponds to the multiplication by the maximum inter-teardrop distance of the series and division by the inter-teardrop distance \( |\mathbf{m}_2 - \mathbf{m}_1| \) in a random radiograph. Here, the maximum inter-teardrop distance is indicated with \( |\mathbf{m}_{20} - \mathbf{m}_{10}| \). It is equated to the inter-teardrop distance at the mean of the variables, which is the distance between the teardrop figures’ image coordinates calculated using Eq. 2 and setting all random variables to zero:

\[
|m_{20} - m_{10}| = f \cdot \left| \begin{array}{c}
\left( \frac{M_{20x}}{M_{20z}}, \frac{M_{20y}}{M_{20z}} \right)^T - \left( \frac{M_{10x}}{M_{10z}}, \frac{M_{10y}}{M_{10z}} \right)^T
\end{array} \right|.
\]
Additionally considering the correction for magnification, the distance vector is (see Fig. 1)

\[
d = \frac{r}{b + \delta b} \left[ \frac{\cos(\omega) \cdot |c - m_2| \cdot |m_{20} - m_{10}|}{\sin(\omega) \cdot |c - m_2|} \right].
\]

The multiplication by \( r/(b + \delta b) \) instead of \( r/b \) takes the variability of determining \( b \) into account. \( \cos(\omega) \) can be obtained by the dot product of \( (c - m_2) \) and \( (m_2 - m_1) \), \( \sin(\omega) \) can be calculated, for example, by application of the formulas for the angle between two straight lines, here given by \( c - m_2 \) and \( m_2 - m_1 \) (see, e.g., (Bronstein and Semendjajew, 1989 p. 220)). The result for the distance vector is

\[
d = \frac{r}{b + \delta b} \left[ \frac{(c - m_2)(m_{20} - m_{10}) + (c - m_2)(m_{21} - m_{11})}{|m_2 - m_1|} \right].
\] (4)

3.2.2. The function \( d \) representing the distance vector in Sutherland et al.

The most caudal point of the teardrop figure next to the prosthesis is indicated with \( m_1 \), the upper point of the Köhler line with \( m_2 \), and the lower point of the Köhler line with \( m_3 \) (see Fig. 2). The distance between \( c \) and the Köhler line can be calculated by inserting the length of the vector \( c - m_3 \) and the projection of \( c - m_3 \) on \( m_2 - m_3 \) into the Pythagoras Theorem. The y-distance is simply the difference of the y-coordinates of \( c \) and \( m_1 \) and the correction for the magnification is the same as in (Nunn et al., 1989) and (Dickob et al., 1994). Thus, the distance vector is

\[
d = \frac{r}{b + \delta b} \left[ \frac{\sqrt{|c - m_3|^2 - (c - m_3)^T(m_2 - m_3)/|m_2 - m_3|^2}}{m_{1y} - c_y} \right].
\] (5)

3.2.3. The function \( d \) representing the distance vector in EBRA

The most caudal image points of the foramina obturatoria are indicated with \( m_1 \) (left) and \( m_2 \) (right). The most medial point of the contour of the foramen obturatum next to the prosthesis and the middle cranial end of the symphysis pubis are indicated with \( m_3 \) and \( m_4 \), respectively (see Fig. 3). The assumed intersection of the vertical beam with the film is given by \( \frac{1}{2} m_4 \), as it is the midpoint between the film center \((0, 0)\) and \( m_4 \).

As in Section 2.4 described, the measured 2D distance corresponds to the distance between the reconstructed 3D position of the implant reference point \( C' \) and an x- and a y-reference-plane with the normals \( n_x \) and \( n_y \). It is given by

\[
d_x = n_x^T \cdot C' \quad \text{and} \quad d_y = n_y^T \cdot C'.
\] (6)

In order to calculate \( C' \), \( n_x \) and \( n_y \) an estimate of the film–focus constellation is required. A minimal and a maximal value for the film–focus distance are calculated inserting \( r, b \) (as expressed by formula 5), and the extremal values for the film–implant distance \( h_{\text{max,min}} \) in the equation

\[
f_{\text{max,min}} = (b + \delta b) \cdot \frac{r}{\sqrt{(b + \delta b)^2 + (h_{\text{max,min}})^2}} + \arctan \left( h_{\text{max,min}} \left( \frac{b + \delta b}{r} \right) \right).
\] (7)

Into the original formula (Krismer et al., 1995, p. 1228), \( b + \delta b \) is inserted instead of \( b \) to take the variability of measuring the radius of the femoral head into account. If \( f_{\text{max}} \) is greater than 1300 mm and \( f_{\text{min}} \) smaller than 900 mm, the film–focus distance is set to 1300 or 900 mm, respectively.

The 3D implant reference point \( C' \) can be reconstructed by re-projecting \( c \) with the assumed position of the focus defined by \( \frac{1}{2} m_4 \) and \( f_{\text{max,min}} \). The implant’s z-position is \( C'_{y_{\text{max,min}}} = f_{\text{max,min}} - h_{\text{max,min}} \). Its x-y-position at \( C'_{y_{\text{max,min}}} \) is given by

\[
C'_{x,y_{\text{max,min}}} = \left( f_{\text{max,min}} - h_{\text{max,min}} \right) \left( c_{x,y} - \frac{1}{2} m_{4x,y} \right). 
\]

The normal of the y-reference-plane \( n_y \) is the normalized cross product of the rays reaching from the assumed position of the focus to \( m_1 \) and to \( m_2 \) (see Fig. 3):

\[
n_y = \frac{\begin{bmatrix} m_{1x} - \frac{1}{2} m_{4x} \\ m_{1y} - \frac{1}{2} m_{4y} \\ f_{\text{max,min}} \end{bmatrix} \times \begin{bmatrix} m_{2x} - \frac{1}{2} m_{4x} \\ m_{2y} - \frac{1}{2} m_{4y} \\ f_{\text{max,min}} \end{bmatrix}}{\left| m_{1x} - \frac{1}{2} m_{4x} \right| \times \left| m_{2x} - \frac{1}{2} m_{4x} \right| - \left| m_{1y} - \frac{1}{2} m_{4y} \right| \times \left| m_{2y} - \frac{1}{2} m_{4y} \right|}.
\] (8)

The x-reference-plane is defined by \( m_3 \) and \( m_4 \), which is the projection of \( m_3 \) on the vector \( m_1 - m_2 \):

\[
m_3' = m_2 + \frac{(m_3 - m_2) \cdot (m_1 - m_2)}{|m_1 - m_3|^2} (m_1 - m_2).
\]

Replacing \( m_1 \) with \( m_1' \) and \( m_2 \) with \( m_3 \) in Eq. (8), the normal of the x-reference-plane \( n_x \) can be calculated analogously to \( n_y \). Now, with the formulas for \( f_{\text{max,min}} \), \( C' \), \( n_x \) and \( n_y \), the 2D distance can be calculated using Eq. (6). The result is:
The principle of error propagation

For the sake of clarity, the insertion of the expression (7) for \( f^{\text{max, min}} \) was omitted. As mentioned in Section 2.4, the final 2D distance is the mean of \( d_{x,y}^{\text{max}} \) and \( d_{x,y}^{\text{min}} \).

### 3.3. The principle of error propagation

Assuming that the random variables \( v_j \) are uncorrelated, the covariance matrix of the random vector \( \mathbf{v} = (v_1, \ldots, v_m)^T = (\Delta \alpha, \Delta \beta, \Delta \gamma, \Delta T_x, \Delta T_y, \Delta T_z, \Delta c_x, \Delta c_y, \Delta m_1, \ldots, \Delta m_m, \Delta v, \ldots, \Delta v_m)^T \) is diagonal and has the form

\[
\Sigma_{vv} = \begin{bmatrix}
\sigma_{v_1}^2 & 0 & \cdots & 0 \\
0 & \sigma_{v_2}^2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sigma_{v_m}^2
\end{bmatrix},
\]

where \( \sigma_{v_j}^2 \) are the variances of the variables \( v_j \). Using the principle of propagation of variances and after linearizing \( d(p, v_j) \) in terms of \( v \), the covariance matrix of the distance vector \( \Sigma_{dd} \) becomes

\[
\Sigma_{dd} = J \Sigma_{vv} J^T
\]

\[
= \left( \sum_j \sigma_{v_j}^2 \left( \frac{\partial d}{\partial v_j} \right)_{v_j = \bar{v}_j} \right)^2 \sum_j \sigma_{v_j}^2 \left( \frac{\partial d}{\partial v_j} \right)_{v_j = \bar{v}_j} \left( \frac{\partial d}{\partial v_j} \right)_{v_j = \bar{v}_j}^T,
\]

where \( J \) is the Jacobian. It is given by the partial derivatives of \( d \) with respect to the variables \( v_j \) at the mean value \( \bar{v} = (0, 0, \ldots, 0)^T \) of \( \mathbf{v} \). The square roots of covariance matrix’ diagonal elements are the standard deviations looked for:

\[
\sigma_d = \sqrt{\sum_j \left( \sigma_{v_j} \frac{\partial d}{\partial v_j} \right)_{v_j = \bar{v}_j}^2},
\]

\[
\sigma_y = \sqrt{\sum_j \left( \sigma_{v_j} \frac{\partial d}{\partial v_j} \right)_{v_j = \bar{v}_j}^2}.
\]

### 4. Application of the scheme and results

The functions for the distance vector (Eqs. (4), (5) and (9)) were entered in Mathematica (Wolfram, 1996). In doing so the expressions for \( c, m_1 \) and \( b \) as given by the Eqs. (1), (2) and (3) were inserted in each function. Thus, the resulting functions for \( d \) were depending only on the random variables \( v_j \) and on the parameters \( C_{ij}, M_{ij}, Z, f, r \) and \( h^{\text{max, min}} \) (Krismer et al., 1995). The partial derivatives \( \partial d_i / \partial v_j \) at \( v = \bar{v} \) were calculated, and the values for the parameters were inserted.

In the following section, the anatomical points defining \( C_{ij}, M_{ij} \) and \( Z \) are described. Their coordinates are listed in Table 1.

The resulting derivatives are a measure for the influence of the respective variable on the precision of the measured migration. They are shown in Sections 4.2–4.4, where values smaller than \( 10^{-5} \) are not listed.

In (Nunn et al., 1989; Dickob et al., 1994; Sutherland et al., 1982), the migration corresponds to the change of the distance vector measured using two different radiographs. The multiplication of \( \sigma_{d_i} \) and \( \sigma_{v_j} \) by \( \sqrt{2} \) is required to finally obtain the standard deviation of the migration. In
order to calculate the final error of the migration in
(Krismer et al., 1995), the effect of the comparability algorithm is considered in Section 4.4.1.

4.1. Assumed 3D coordinates of the implant and of the bony landmarks

Calculating the partial derivatives of the \( x-y \)-distance requires the 3D coordinates of the following points: the most caudal and the most medial points of the foramina arcuata and of the ipsilateral foramen obturatum. The mean orientation is taken to be identical to the most lateral point of the ipsilateral linea arcuata and of the ipsilateral foramen obturatum. The center of the femoral head was taken as 3D implant reference point. The rotation center was assumed to lie on the hip’s midline. The mean \( y \)- and the mean \( z \)-coordinates of most dorsal points (left and right side) of the sacrum were chosen as \( y \)- and as \( z \)-coordinates of the rotation center.

4.2. The standard deviation of the distance measured in (Nunn et al., 1989) and (Dickob et al., 1994)

The standard deviation of the distance vector is

\[
\sigma_{\Delta r} = (0.07218\sigma_{r_0})^2 + (0.01070\sigma_{r_0})^2 + (-0.00360\sigma_{r_0})^2 \\
+ (0.00342\sigma_{r_1})^2 + (0.4955\sigma_{r_1})^2 + (0.74423\sigma_{r_1})^2 \\
+ (-0.93769\sigma_{r_1})^2 + (-0.08146\sigma_{r_1})^2 + (0.19346\sigma_{r_1})^2 \\
+ (0.08146\sigma_{r_1})^2 + (-1.55878\sigma_{r_0})^2 + (0.01231\sigma_{r_0})^2. \\
\sigma_{\Delta z} = (0.04555\sigma_{z_0})^2 + (0.00002\sigma_{z_0})^2 + (-0.00342\sigma_{z_0})^2 \\
+ (-0.00001\sigma_{z_0})^2 + (-0.74423\sigma_{z_0})^2 + (0.93769\sigma_{z_0})^2 \\
+ (-0.19346\sigma_{z_0})^2 + (-0.65634\sigma_{z_0})^2 + (0.00001\sigma_{z_0})^2. \\
\]

For both components the significance of pelvis rotations is rather small because of the short 3D distance between implant reference point and bony landmarks. The little effect of translations along the \( x \)-axis on \( \sigma_{\Delta y} \) and along the \( y \)-axis on \( \sigma_{\Delta z} \) results from the only slightly different heights of \( C_0 \) and \( M_{10} \).

---

In contrary to the general expectations, the correction of the x-distance for rotation around the y-axis deteriorates the measurement: the correction is a multiplication of the measured x-distance by the quotient \(\frac{|m_{10}-m_{20}|}{|m_1-m_2|}\), where the numerator is constant. This quotient is supposed to reflect the cosine of the rotation angle \(\delta \beta\) but the proportional change of \(d\), at a y-rotation. That would be the case for parallel projections (great film–focus distance). In a standard X-ray system with \(f = 900\) mm, however, the inter-teardrop distance does not only depend on \(\delta \beta\) but is also influenced by the heights of the teardrop figures changed by the rotation. Therefore, even if pelvis rotation around the y-axis was the only source of error, the correction for magnification would rather lead to a biased measurement than neutralize the rotation’s influence.

Apart from that, the inter-teardrop distance \(|m_1-m_2|\) depends on the orientation relative to the x-axis, on the z-position of the pelvis, and on the film–focus distance, because all these variables affect the z-distance of the teardrop figures to the focus. Thus, the multiplication by \(\frac{|m_{10}-m_{20}|}{|m_1-m_2|}\) strongly increases the standard deviation. Omitting the correction for rotation yields \(\frac{\partial d_x}{\partial \delta \alpha} = 0.02337\), \(\frac{\partial d_x}{\partial \delta T_y} = -0.00030\) and \(\frac{\partial d_y}{\partial \delta T_y} = -0.00030\) whereas considering the correction increases these partial derivatives up to the values listed above.

4.3. The standard deviation of the distance measured in (Sutherland et al., 1982)

Here, \(\sigma_{d_x}\) mainly depends on the rotation around the z-axis:

\[
\sigma_{d_x}^2 = (0.01579\sigma_{i_1})^2 + (-0.12875\sigma_{i_0})^2 + (-0.01030\sigma_{d_y})^2 + (0.00979\sigma_{d_y})^2 + (0.00126\sigma_{d_y})^2 + (-0.00700\sigma_{d_y})^2 + (0.78433\sigma_{d_y})^2 + (0.10073\sigma_{d_y})^2 + (-0.31904\sigma_{d_y})^2 + (-0.04097\sigma_{d_y})^2 + (-0.46529\sigma_{d_y})^2 + (-0.05976\sigma_{d_y})^2 + (-1.59343\sigma_{d_y})^2 + (-0.00070\sigma_{d_y})^2,
\]

\[
\sigma_{d_y}^2 = (-0.04432\sigma_{i_2})^2 + (0.00280\sigma_{i_0})^2 + (-0.58407\sigma_{d_y})^2 + (-0.00264\sigma_{d_y})^2 + (-0.00001\sigma_{d_y})^2 + (-0.79078\sigma_{d_y})^2 + (0.79078\sigma_{d_y})^2 + (-0.69730\sigma_{d_y})^2 + (-0.00001\sigma_{d_y})^2.
\]

The great value for \(\partial d_y/\partial \delta y\) results from the fact that \(d_y\) – as far as it is described in (Sutherland et al., 1982) – just corresponds to the difference of the y-coordinates \(c_y - m_{1y}\), and the rotation around the z-axis is neglected. Translations and x-y-rotations have little influence on \(\sigma_{d_y}\) as well as on \(\sigma_{d_z}\) because of the short z-distance of the teardrop figure and of the anatomical points defining the Köhler line to the 3D implant reference point.

4.4. The standard deviation of the distance measured in EBRA

In (Krismer et al., 1995), the x-y- as well as the z-distances between the 3D implant reference point and the 3D bony reference structures are greater than in the other methods. Therefore, the migration is more sensitive towards the rotation of the pelvis. The great \(\sigma\)-distances are also the reason for the large values for \(\partial d_x/\partial \delta T_x\) and \(\partial d_y/\partial \delta T_x\):

\[
\sigma_{d_x}^2 = (-0.00657\sigma_{i_1})^2 + (-0.24318\sigma_{i_0})^2 + (-0.01057\sigma_{d_y})^2 + (0.01004\sigma_{d_y})^2 + (0.03702\sigma_{d_y})^2 + (-0.79479\sigma_{i_1})^2 + (0.22159\sigma_{d_y})^2 + (-0.22159\sigma_{d_y})^2 + (0.79730\sigma_{i_1})^2 + (-0.00123\sigma_{d_y})^2 + (1.60297\sigma_{d_y})^2 + (0.00216\sigma_{d_y})^2 + (-0.00747\sigma_{i_1})^2.
\]

\[
\sigma_{d_y}^2 = (-0.38049\sigma_{i_2})^2 + (-0.03987\sigma_{i_0})^2 + (-0.2064\sigma_{d_y})^2 + (-0.02278\sigma_{i_1})^2 + (-0.79516\sigma_{i_1})^2 + (1.11081\sigma_{i_1})^2 + (-0.31668\sigma_{d_y})^2 + (-0.00051\sigma_{d_y})^2 + (-0.99259\sigma_{i_1})^2 + (-0.00051\sigma_{d_y})^2 + (-0.00436\sigma_{d_y})^2.
\]

As the calculated value for the minimal film–focus distance felt short of the predefined limit, the fixed value \(f_{min} = 900\) mm had to be inserted in the formula for \(d_{min}\) (see Sections 2.4 and 3.2.3). This corresponds to setting the ratio between film–focus and implant–focus distance to a constant independent from the radiograph. It is responsible for the high sensitivity of \(d_{xy}\) towards pelvis translation in z-direction and towards the variability of the film–focus distance.

An adapted estimation of the film–focus distance would diminish the influence of the variable orientation. The derivatives with respect to the rotation angles would be (mainly) reduced to \(\partial d_x/\partial \delta \alpha = -0.02105\), \(\partial d_x/\partial \delta \beta = -0.17759\), \(\partial d_y/\partial \delta \alpha = -0.36131\) and \(\partial d_y/\partial \delta \beta = -0.00010\) if both \(f_{max}\) and \(f_{min}\) were set to the values calculated using Eq. (7) and reflecting the actual magnification. This effect can also be seen in (Nunn et al., 1989; Dickob et al., 1994), where the correction for magnification also decreases these partial derivatives.

On the other hand, setting the ratio between film–focus and implant–focus distance to a constant makes the measurement less sensitive towards the error in determining the radius of the femoral head. The partial derivative with respect to \(\delta b\), which generally is approximately proportional to the length of the measured distance, is reduced by a factor 2. While in the other methods \((\partial d_{xy}/\partial \delta b)/d_{xy}\) is about 1/20, here it is only about 1/40.

The migration measurement according to Krismer et al. (1995) is much more sensitive towards pelvis rotation than
the migration measurement according to Nunn et al. (1989). However, this sensitivity may be compensated by the restriction of the distance measurement to comparable radiographs.

4.4.1. The effect of the comparability algorithm

For simplicity, only the \( x \)-migration is regarded. However, the following considerations are equally valid for the \( y \)-migration.

In order to estimate the effect of the comparability algorithm, a follow-up study of four exposures made at the times \( t_0 \), \( t_1 \), \( t_2 \) and \( t_3 \) after implantation was assumed. According to Krismer et al. (1995) this is the minimum number to determine migration. Additionally, it was assumed that the series contains two pairs of comparable radiographs, and that the time interval lying between the exposure dates of one pair overlaps with the time interval of the other one (see Fig. 5). Usually, the exposures are taken with growing time intervals. Based on the exposure dates in a migration study using EBRA (Krismer et al., 1996) \( t_0 = 1.5 \) months, \( t_1 = 3 \) months, \( t_2 = 12 \) months and \( t_3 = 24 \) months was chosen.

In the following paragraphs \( d_i \) (\( i = 0, \ldots, 3 \)) denotes the \( x \)-distance measured using the radiograph acquired at the time \( t_i \). For all \( d_i \), \( \sigma_{d_i} \) is assumed to be the same. It has to be calculated inserting the variances of the rotation angles reduced a posteriori by evaluating only comparable radiographs.

Three cases of determining the migration are distinguished: (a) only one pair of comparable radiographs is available; (b) the migration during the time where the intervals of two pairs overlap supposed to be measured (here the time between \( t_1 \) and \( t_3 \)); (c) finding out the total migration of the follow-up series is intended.

In case (a), the migration is simply the difference between the 2D distances in two radiographs, and its standard deviation is \( \sqrt{2} \cdot \sigma_{d_i} \), like in the other methods. In case (b), two values for the migration are measured using two pairs of comparable radiographs, and the mean is calculated. The mean migration \( d_{x_2} - d_{x_1} \) between \( t_1 \) and \( t_2 \) is

\[
\frac{d_{x_2} - d_{x_1}}{2} = \frac{1}{2} \left( \frac{t_2 - t_1}{t_2 - t_0} (d_{x_2} - d_{x_0}) + \frac{t_2 - t_1}{t_3 - t_1} (d_{x_3} - d_{x_1}) \right).
\]

Applying the principle of error propagation on the above equation, the standard deviation of the migration between \( t_1 \) and \( t_2 \) is \( 0.5 \sqrt{2} \cdot 0.857^2 + 2 \cdot 0.429^2 \cdot \sigma_{d_i} = 0.68 \cdot \sigma_{d_i} \). In case (c), the migration is determined by adding portions of the migration of the two comparable pairs. Thus, for example, the migration \( d_{x_3} - d_{x_0} \) between \( t_0 \) and \( t_3 \) is given by the following equation:

\[
d_{x_3} - d_{x_0} = 0.5(t_2 + t_1) - t_0 \frac{d_{x_2} - d_{x_0}}{t_2 - t_0} + \frac{t_3 - 0.5(t_2 + t_1)}{t_3 - t_1} (d_{x_3} - d_{x_1}).
\]

Again applying the principle of error propagation, the standard deviation for the total migration is \( \sqrt{2} \cdot 0.571^2 + 2 \cdot 0.786^2 \cdot \sigma_{d_i} = 1.37 \cdot \sigma_{d_i} \).

5. Discussion

In this study a scheme to objectively estimate the error of measuring 2D implant migration using standard radiographs was worked out. Applying the scheme yields an expression for the standard deviation of the 2D migration measurement method under study. Additionally, it provides measures for the sensitivity of the method towards the sources of error. The resulting general scheme can be used to identify the critical sources of error in all cases of measuring distances in 3D to 2D projections.

Four previously published methods for measuring the migration of the prosthetic cup have been analysed. The entering errors considered have been the variability of the intrinsic and extrinsic parameters and the error in locating image structures. For the sake of clarity, distortion and scale of the X-ray image (e.g., arising from scanning the radiograph) and the deformation of the bone have been neglected. However, these additional sources of error could be easily involved into the scheme.

In order to gain a first insight into the error to be expected, the final error of the migration was calculated using the resulted expressions. This requires the standard deviations of the variables describing the sources of error. Up to now, the magnitudes of the entering errors have been estimated only partly and empirically. As a more reliable and complete determination of the error amounts goes
The pelvis orientation with respect to the z-axis has not been considered in the literature up to the present. Its standard deviation was estimated to be $\sigma_{gb} = 3.0^\circ$.

6. Outlook

Based on the results of this study, 2D migration measurement can be improved in two ways: reducing the sensitivity towards the entering errors (see the first two of the following paragraphs), or reducing the magnitude of the entering errors (see the last two paragraphs). As proposed here, the latter mainly concerns the minimization of the error in locating the bony landmarks and the implant in the X-ray image.

Considering the significance of the variable orientation and position of the pelvis in the X-ray unit, in an improved method bony reference points with a short 3D distance to the implant should be chosen. Especially the sagittal component of this 3D distance is relevant. Using a bony landmark close to the implant strongly decreases the influence of the pelvis’ horizontal translation and rotation around the medio-lateral and the cranio-caudal axis between two exposures. On the other side, a correction for this rotation by just comparing the varying distances between anatomical structures in the radiographs of the follow-up study should be avoided. An influence on the measurement of rotation around the sagittal axis can be excluded using an image coordinate system defined by a line connecting two bony landmarks. The measured 2D distance should be corrected for magnification estimating the ratio between film–focus and implant–focus distance in each radiograph. Predetermined values for this ratio should not be inserted. First, the correction for magnification makes the measurement independent from the varying film–focus and pelvis–focus distance. Second, it also can decrease the sensitivity of the measured migration towards the variable orientation of the pelvis. As in case of an adapted correction for magnification none of the intrinsic parameters affects the measured 2D migration, a reconstruction of the film–focus constellation at exposure is not necessary.

One possibility to reduce the error in locating the bony...
landmarks is the application of a template matching algorithm. Using such an algorithm, the 2D translation between the template, i.e., the area containing the landmark in a reference radiograph, and the area of the landmark in the radiograph under study theoretically can be estimated with sub-pixel accuracy (Danuser and Mazza, 1996; Berger and Danuser, 1997). The position of the bony landmarks in the evaluated X-ray images can be yielded by adding the estimated translations to an arbitrarily chosen but constant reference point in the template. Because of an automated estimation of the 2D translation, template matching minimizes the inter- and intra-observer variability of locating bony landmarks. In order to optimize the location of the implant and to precisely determine the magnification factor, the 3D information given by the CAD model of the prosthetic cup can be used. Knowing the surface, the radiographic image of the cup can be simulated. The implant’s 3D position and orientation in the X-ray system can be estimated by applying the principle of analysis-by-synthesis: the minimization of the difference between original and simulated image by adapting the position and orientation parameters. Thus the magnification factor as well as the image reference point of the implant, i.e., the projection of its estimated 3D reference point on the radiograph, can be determined precisely.

The variability of the pelvis orientation can be decreased either by controlling the rotation of the pelvis at exposure, e.g., as described by Kirkpatrick et al. (1983), or by the retrospective limitation of the rotation excluding radiographs where the pelvis is strongly rotated, like in EBRA. However, the comparability criterion described in (Krismer et al., 1995) seems to be inadequate, because image distances depending on all variables, not only on the rotation angles, are used to estimate the rotation between two radiographs. A more reliable criterion could be given by the implant’s orientation parameters estimated using analysis-by-synthesis.

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