Numerical modeling of heart valves using resistive Eulerian surfaces

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SUMMARY

The goal of this work is the development and numerical implementation of a mathematical model describing the functioning of heart valves. To couple the pulsatile blood flow with a highly deformable thin structure (the valve’s leaflets), a resistive Eulerian surfaces framework is adopted. A lumped-parameter model helps to couple the movement of the leaflets with the blood dynamics. A reduced circulation model describes the systemic hemodynamics and provides a physiological pressure profile at the downstream boundary of the valve. The resulting model is relatively simple to describe for a healthy valve and pathological heart valve functioning while featuring an affordable computational burden. Efficient time and spatial discretizations are considered and implemented. We address in detail the main features of the proposed method, and we report several numerical experiments for both two-dimensional and three-dimensional cases with the aim of illustrating its accuracy. Copyright © 2015 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The cardiovascular system possesses a very complex structure that can be characterized by different physical scales in space (ranging from micrometers for blood cells to centimeters for organs) as well as in time (e.g., milliseconds for chemical-driven processes and order of tens of seconds for the systemic circulation). By means of rhythmic contractions, the heart provides a continuous circulation of blood to the entire body. The heart is made of two joint parts, called the left heart and the right heart, respectively; it has four chambers consisting of two superior atria and two inferior ventricles. At the exit of each heart cavity, the cardiac flow is controlled by four valves featuring stiff but compliant flaps of tissue of similar size (leaflets), which basically prevent backflow between the chambers. The particular aortic valve consists of three crescent-shaped leaflets, faced by three corresponding pouches of the aorta called sinus of Valsalva, which conform the aortic root. In the healthy case, leaflets synchronize the valve opening during systole and their closing during diastole, leading, respectively, to the correct blood ejection and prevention of regurgitation. In contrast, the mitral valve has a bicuspid shape and a more complex anatomy characterized by biological attachments that prevent prolapse during the closure phase thanks to the chordae tendineae. The life quality may drastically deteriorate when valves are dysfunctional. The main valvular pathologies include regurgitation and stenosis.

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Potentially, the efficient and accurate computational investigation of processes related to heart valves functioning may greatly help in tackling specific problems of clinical relevance and in enhancing developments of prosthetic heart valves. Nevertheless, this is a tremendously challenging task. A crucial difficulty is due to the fact that the recovery of motion patterns of blood and valves entails the resolution of a transient fluid–structure interaction (FSI) problem, where the structural properties of the tissue and the blood flow motion are tightly coupled. From the viewpoint of their anatomic features, heart valves can be regarded as thin elastic structures immersed in a fluid. The elastic properties of the leaflets are clinically relevant but difficult to model. Bending resistant forces exist within the valve, which are controlled by the internal and inhomogeneous fiber structure of the leaflets [1, 2]. Full three-dimensional realistic models are still computationally demanding [3]. Moreover, topology changes occur at the closure of the valve, which immediately rule out a wide set of numerical methods as candidates to describe the motion of the structure. The large deformation of the leaflets during the opening and closing phases induces the necessity of introducing efficient mesh adapting strategies [4]. Moreover, high pressure discontinuities occur along the leaflets, especially when the valve is closed, where they undergo a significant physiological pressure load [5]. All these aspects highlight the difficulties in modeling the interaction between leaflets mechanics (including contact, coaptation, and chordae tendineae plus papillary muscles for the mitral/tricuspid valves) and blood dynamics.

Reduced order modeling is a useful formalism that can provide (partial) information on the cardiovascular response. Important developments have appeared exploiting geometrical multiscale techniques, in which zero-dimensional and one-dimensional models are coupled with multidimensional descriptions of the cardiovascular system [6–10]. The modeling procedure typically consists in local investigations of the hemodynamics using detailed three-dimensional models and geometries, whereas boundary conditions are provided by their coupling with zero-dimensional or one-dimensional models [11]. Pressure and flow rate may be directly exchanged between models of different geometrical dimension [12, 13]. Applications to the modeling of aortic flow are presented in [14–20]. Multiscale models have been widely used to model valves but mainly as a boundary condition for multidimensional blood flow simulations. Concerning lumped (i.e., zero-dimensional) models of heart valves, the first strategies considered valves as diodes plus a linear or nonlinear resistance; see, for example, [21]. Other strategies employ a perfect diode model that allows the blood to flow in one specific direction of the circulatory system, preventing any backflow [16]. Improvements of that model were achieved by adding a small resistance to the flow that mimics the open valve in normal physiological conditions [22]. All these models describe valves in their either fully open or fully closed position and neglect the dynamics of the opening and the closing. More physiologically realistic studies considered blood–leaflets interactions, addressing the effect of pressure difference and shear stress in the valve dynamics but omitting the vortical phenomena in the sinuses of Valsalva [3, 23, 24]. Since then, more accurate reduced order models introduced intermediate models between fully three-dimensional models and simplified diode-like valve models by including lumped parameters describing the interaction between blood and leaflets [25–27].

Typically, multidimensional models for the simulation of heart valves are much more demanding from the viewpoint of computational cost and model complexity. They can be roughly sorted according to the way they address the FSI coupled problem: structural, fluid, or fluid–structure solvers.

Regarding the first group, the mechanical response of valves during the opening and closure phases was investigated in [28], while the effect of blood is modeled by a transvalvular pressure drop. Models of mitral valve focusing mainly on the pathological case of ischemic mitral regurgitation were presented in [2, 29]. These studies foresee hyperelastic anisotropic mechanical properties for the mitral valve tissue, and a physiological pressure load was applied to the valve to simulate valve closure until peak systole. Transversely isotropic hyperelastic models such as those introduced in [30] were employed in [31] to perform structural analysis of the mitral valve.

In the second group, the driving idea, following the seminal work of Peskin [32] on the immersed boundary method (IBM), is that a purely fluid-based solver can be employed to describe the valves dynamics. This is typically achieved by replacing the valves by a set of immersed surfaces acting as a resistance in the fluid [5]. In this case, the intrinsic mechanical behavior of the leaflets is neglected,
allowing a drastically reduced computational cost, and an increase in robustness and stability with respect to classical FSI models [5, 33–35]. In [36, 37], the IBM was employed to model the opening and closing dynamics of the aortic valve, while considering the fiber elasticity of the leaflets.

Concerning the latter group, FSI solvers opt for following the moving boundaries in a Lagrangian description, keeping a separate (Eulerian) model for the fluid and coupling them in a suitable manner. Different versions of such a formalism are reported in [3, 38–43]. The contact between leaflets during the closing phase is sometimes handled using a penalization approach; see, for example, [44]. FSI models for the closing of the mitral valve have also been developed considering a compressible fluid [45, 46]. There, non-constant tissue thickness of the mitral valve and nonlinear anisotropic material properties for the valve were considered, including also the collagen fiber orientation and the chordae tendineae. Recent computational studies using subject-specific geometries of the mitral valve [47, 48] highlight its significant influence on the numerical results.

In this paper, we focus on the development and its efficient implementation of a mathematical model of the valves based on the resistive Eulerian surface formalism. The proposed methodology can be somehow placed in the second group described earlier, as we opt for an Eulerian description that offers a compromise between lumped-parameter models and fully three-dimensional FSI problems. Our approach has the advantage of allowing a three-dimensional description of the leaflets at a reasonable computational cost compared with the classical FSI formulation. It is more general than that derived in [5] as far as the opening and closing processes of the valve are modeled. The Eulerian description enables to readily capture the valve motion, while remaining less challenging in terms of meshing problems. Moreover, this strategy allows to easily handle the contact between the leaflets and to capture the transvalvular pressure drop when the valve closes. Furthermore, the extension of this framework to the patient-specific study is relatively straightforward.

Our objectives are to simulate blood flow patterns in the aortic root and in the ascending aorta and to study how these flow features interact with the compliant leaflets.

We have arranged the remainder of this paper as follows. Section 2 outlines the main aspects of the model derivation, including the representation of the aortic valve as an open thin structure, the reduced order representation of the valve motion via its opening angle, the Navier–Stokes equations governing the blood dynamics, the level-set formulation to capture the position of each leaflet, and the boundary conditions along different boundaries, which will allow us to recover a realistic valve motion when prescribing physiological pressure waveforms.

2. DERIVATION OF THE MODEL PROBLEM

In this section, we introduce the mathematical model describing the motion of the three-dimensional leaflets. In the first part, we model leaflets as thin surfaces immersed in an incompressible viscous fluid. Then, we present the fluid problem and the terms required to describe the movement of the thin valve immersed in blood without using a surface mesh fitting the leaflet geometry. Finally, we specify the boundary conditions imposed along different boundaries, which will allow us to recover a realistic valve motion when prescribing physiological pressure waveforms.

2.1. On the modeling of heart valves

The accurate description of ventricular flow is complex, and efficient computational strategies are essential to model blood flow patterns, especially in the surrounding of valves. After reviewing some modeling strategies, we will introduce the model we employ to describe the leaflets, which in particular circumvents some numerical difficulties typically associated with Lagrangian descriptions of motion.
2.1.1. **Open surface description.** We first emphasize that the passive and active mechanical properties of the leaflets are not modeled within the present work. The geometrical approximation of valves has been typically based on either Lagrangian or Eulerian approaches. In Lagrangian-based frameworks, usually a mesh fitting the leaflets’ geometry is employed following the valve motion. In such cases, major complications are due to the need of consistently remeshing at each time step and interpolating the solution between different meshes. In Eulerian-based approaches, leaflets can be implicitly defined using a single mesh for the whole computational domain (fluid and valve). In this way, one does not need to resort to meshes with immersed two-dimensional elements (triangle facets) following the valve surface or to introduce a fictitious thickness of the leaflets if volumic meshes are used. In our approach, leaflets are described as thin structures mimicking the real geometry, but without considering their elastic properties. More precisely, we consider each leaflet to be an open surface of finite extent in a three-dimensional volume and attached to the aortic root at the lower Valsalva sinuses; see Figure 1 (left–middle). In this way, each leaflet represents a two-dimensional manifold with a one-dimensional boundary embedded within a three-dimensional space in such a way that the boundary of the manifold lies completely within a bounded volume. The open surface is defined as a co-dimension one of a manifold with boundary embedded in three-dimensional Euclidean space, whereas closed surfaces, as, for example, spheres, can be regarded as similar manifolds without boundary. The closed surfaces can be also described implicitly as the set of solution points to a scalar equation \( \phi(x) = 0 \), where \( x \in \mathbb{R}^3 \). We assume that the leaflets shape in both open and closed positions of the valve can be accurately approximated by polynomials. The surface is called an algebraic surface of order \( p \) if the implicit function \( \phi \) is a polynomial of degree \( p \), and we name surface order the order of the polynomial function \( \phi \).

We will employ second-order algebraic surfaces because linear functions would not be appropriate to describe both the open and closed positions of the leaflets. At each time \( t \), the surface is defined by

\[
\phi(t; x) = \sum_{0 \leq i + j + k \leq 2} \alpha_{i,j,k} x^i y^j z^k = 0, \quad x = (x, y, z) \in \mathbb{R}^3.
\]

Using anatomical measurements taken from the existing literature [49], a set of control points helps to parametrize the geometry, and the coefficients of the polynomial are obtained by fitting these landmarks. The dimensional parameters for the geometry of the leaflets are reported in Table II.

Closed surfaces can be generally represented in terms of level sets, but this seems practically impossible for open surfaces. We therefore propose a construction based on two level-set embedding functions describing each leaflet at a given time. The advantage is that we do not need to move the leaflets through advection of the level-set functions, but we describe their motion as the combination of two level-set embedding functions: a primary level set \( \phi \) representing a closed surface, where
Table I. Set of parameters used in the lumped model (2.1) describing the movement of the aortic valve.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_{\text{max}}$</td>
<td>75</td>
<td>degrees</td>
</tr>
<tr>
<td>$\theta_{\text{min}}$</td>
<td>5</td>
<td>degrees</td>
</tr>
<tr>
<td>$K_f/I$</td>
<td>50</td>
<td>s$^{-1}$</td>
</tr>
<tr>
<td>$K_p/I$</td>
<td>4.1253</td>
<td>rad cm$^2$/s$^2$ dyn</td>
</tr>
<tr>
<td>$K_q/I$</td>
<td>2</td>
<td>rad/s cm$^2$</td>
</tr>
<tr>
<td>$K_v/I$</td>
<td>7</td>
<td>rad/s cm$^2$</td>
</tr>
</tbody>
</table>

The extreme angles $\theta_{\text{max}}$ and $\theta_{\text{min}}$ correspond to those of a healthy valve.

aThree-dimensional case.

a part of it represents the leaflet surface, and a secondary level set $\psi$, which helps to localize the portion that corresponds to the leaflet; see Figure 1 (right). Each leaflet $\Gamma$ (of infinitesimal thickness) is described as follows:

$$\Gamma(t) = \{x(t) \in \Omega : \varphi(t; x) = 0 \text{ and } \psi(t; x) \leq 0\}, \quad \forall t \in (0, T).$$

2.1.2. Lumped-parameter model for the valve movement. In the present paper, we employ a lumped-parameter model to describe the movement of the aortic valve. Following [25, 26, 50], we introduce a differential equation governing the leaflet dynamics as function of the opening angle and based on the effect of blood flow around the valve (i.e., the values of pressure and flow rate). Notice that the movement of the leaflet also has a feedback on the pressure and the local flow rate changes.

This reduced model describes the angular momentum balance in the aortic valve. It balances the angular acceleration of the leaflets with various angular momenta that affect the valve motion, owing to the following: the friction from neighboring tissue resistance at the valve root, the pressure difference across the valve, the dynamic motion of the blood acting on the leaflets, the downstream vortex formation, and the shear stress on the valve. The latter effect is very small (e.g., [24]) and therefore is neglected. Based on clinical and experimental observations, this model assumes that the vortex effect only influences the valve closing process. A more detailed discussion is provided in [51]. Let $K_f$, $K_p$, $K_q$, and $K_v$ be the coefficients respectively corresponding to the different angular momenta described earlier. The momentum of inertia of the valve is denoted by $I$. A detailed description of the model is available in [25, 50, 51]. The needed parameters characterizing the aortic valve model are given in Table I. The following equation drives the aortic valve motion:

$$I \frac{d^2\theta}{dt^2} + K_f \frac{d\theta}{dt} = K_p \left(P_{lv} - P_{ao}\right) \cos(\theta) + K_q \left(P_{lv} - P_{ao}\right) \sin(\theta) + K_v \frac{P_{lv} - P_{ao}}{\text{max}} \cos(\theta),$$

(2.1)

Here, $P_{lv}$ and $P_{ao}$ represent the left ventricular pressure and the aortic pressure, respectively. $Q$ denotes the volumetric flow rate passing through the valve. The non-binary state of the valve is described through a scalar parameter $\xi$, which depends on the rotating angle $\theta$ of valve leaflet, as illustrated in Figure 1. It is given by the following:

$$\xi(\theta) = \frac{(\cos(\theta_{\text{min}}) - \cos(\theta))^2}{(\cos(\theta_{\text{min}}) - \cos(\theta_{\text{max}}))^2}, \quad \theta \in [\theta_{\text{min}}, \theta_{\text{max}}].$$

The minimal angle $\theta_{\text{min}}$ corresponds to the fully closed leaflet position, whereas the maximum opening angle $\theta_{\text{max}}$ corresponds to the fully open leaflet position. Model (2.1) allows to readily recover different pathological scenarios including regurgitation and valve stenosis. For regurgitation, it suffices to consider a large value for $\theta_{\text{min}}$, while stenosis can be modeled by restricting the maximum opening angle $\theta_{\text{max}}$; see Section 4.6. Regarding the coupling of (2.1) with the fluid problem, we...
employ the pressure difference across the leaflet as a condition to manage the opening of the valve, if the valve is closed. However, for an open valve, the leaflets remain in their positions as long as a positive flow rate is generated in the direction of the valve opening. The closure of the valve is instead dictated by a negative value of this flow rate. Algorithm 1 outlines the solution strategy. Notice that the angular position of the valve is computed from (2.1) if the function \texttt{moveValve} returns the Boolean value 1.

Algorithm 1 Updating of the leaflet position
1: Function\{\texttt{moveValve}=0\}: Boolean
2: \textbf{if} (\xi == 0) and (P_{lv} - P_{ao} > 0) \textbf{then}
3: \hspace{1em} return 1;
4: \textbf{end if}
5: \textbf{if} (\xi == 1) and (Q < 0)\textsuperscript{a} \textbf{then}
6: \hspace{1em} return 1;
7: \textbf{end if}
8: \textbf{if} (0 < \xi < 1) \textbf{then}
9: \hspace{1em} return 1;
10: \textbf{end if}
11: \textbf{EndFunction}

\textsuperscript{a}Remark that \(Q > 0\) is the convention when blood is flowing from the left ventricle.

2.2. Problem statement

The mathematical formulation of the problem at hand follows that proposed in [52] to model immersed stents designed for the treatment of cerebral aneurysms, and also used in [5] to describe the open/closed valves. There, each surface was associated with a different resistance value during systole and diastole: the assigned resistance to the closed valve switches from zero to a high value when the valve closes and vice versa when the valve opens. Analogously, the resistance assigned to the open position is defined. Here, we extend that approach to describe the movement of the leaflets. We stress that this methodology also provides a simple and accurate way of computing pressure discontinuity across the valve, independently of the spatial discretization method. A level-set formulation is used to describe the leaflets, which are moved using the low order model that predicts the valve opening angle. The leaflet surfaces are hence equipped with a resistance; that is, they resemble walls not allowing the crossing of fluid through them.

2.2.1. Direct formulation of the model. Fully developed turbulence is rarely observed in healthy humans [53], and the laminar behavior of blood can be assumed because we focus on large vasculature; see, for example, [5, 36]. The Navier–Stokes equations represent a reasonable approximation of the blood flow as a homogeneous, incompressible, and Newtonian fluid.

For \(T > 0\) and for any \(t \in (0, T)\), let \(\Omega(t) \subset \mathbb{R}^3\), \(d = 2, 3\), represent the computational domain occupied by the blood and the valve and having a Lipschitz continuous boundary. Let \(n\) denote the unit outward normal vector on \(\partial\Omega(t)\). The aortic valve is composed by three leaflets \(\Gamma_i(t) \in \Omega(t), i \in \mathcal{N}\) (Figure 1), where \(\mathcal{N} = \{1, 2, 3\}\) is the set of leaflets. As explained in Section 2.2.2, each leaflet \(\Gamma_i(t)\) represents an open surface described by two embedded level-set functions \(\varphi_i\) and \(\psi_i\) such that

\[
\Gamma_i(t) = \{x(t) \in \Omega(t) : \varphi_i(t, x) = 0 \quad \text{and} \quad \psi_i(t, x) \leq 0\}, \quad \forall i \in \mathcal{N}.
\]

Regarding the interaction between the blood flow and the moving valve, a no-slip condition is assumed on the surface of the leaflets. Let \(u\) and \(u^*\) denote the flow velocity and the leaflets velocity, respectively. Hence, the condition \(u = u^*\) on \(\Gamma_i, i \in \mathcal{N}\) holds.

The resistive immersed surface introduces an additional dissipative term in the momentum balance equation, and a Dirac measure \(\delta_{\Gamma_i}, i \in \mathcal{N}\), allows to localize the leaflets, where \(R\) represents their resistance to the flow (i.e., \(R\) can be regarded as a penalization). When the resistance is very

large, the leaflets behave as a solid, and the overall flow velocity is enforced as \( \mathbf{u} = \mathbf{u}^* \) on the leaflets, as already noticed. At the numerical level, the assignment of small values to the resistance helps to model the valve as a porous thin structure, whereas the use of large values of \( R \) may induce an ill-conditioned linear system. The modified Navier–Stokes equations with the resistive terms mimicking the valve read as follows:

Find the fluid velocity \( \mathbf{u} = \mathbf{u}(t, \mathbf{x}) \) and the pressure \( p = p(t, \mathbf{x}) \) such that

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) - \nabla \mathbf{p} + \sum_{i \in N} \delta_{\Gamma_i} R (\mathbf{u} - \mathbf{u}^*) = 0 \quad \text{in} \ (0, T) \times \Omega, \tag{2.2}
\]

\[
\mathrm{div} \mathbf{u} = 0 \quad \text{in} \ (0, T) \times \Omega, \tag{2.3}
\]

where \( \mathbf{\sigma}(\mathbf{u}, p) = 2\mu \mathbf{D}(\mathbf{u}) - \mathbf{p} \mathbf{I} \) is the Cauchy stress tensor, \( \mathbf{D}(\mathbf{u}) = (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2 \) is the infinitesimal strain tensor, and \( \mathbf{I} \) is the identity tensor. We assume constant density \( \rho \) and viscosity \( \mu \). For any leaflet \( \Gamma_i(t), i \in N \), the interface conditions enforce the continuity of the velocity across \( \Gamma_i \), whereas the jump of the normal stress is given by \( \mathbf{\sigma} \mathbf{n} = -R (\mathbf{u} - \mathbf{u}^*) \). Remark that the discontinuity of the normal stress can be calibrated according to the resistance. Problem (2.2)–(2.3) is endowed with initial conditions; we assume that \( \partial \Omega(t) \) is divided into two parts \( \Gamma_D \) and \( \Gamma_N \) (to be specified later on) on which Dirichlet and Neumann boundary conditions are assigned, respectively. A detailed description of initial and boundary conditions will be provided in Section 2.3. Let us introduce the following space of admissible velocities and pressures:

\[
\mathbb{V}(\mathbf{u}_b) = \left\{ \mathbf{v} \in (H^1(\Omega))^d : \mathbf{v} = \mathbf{u}_b \text{ on } \Gamma_D \right\}, \quad \mathcal{Q} = \left\{ q \in L^2(\Omega) : \int_\Omega q \mathrm{d} \mathbf{x} = 0 \right\}.
\]

Testing (2.2)–(2.3) against suitable functions and integrating by parts, the variational formulation reads as follows:

\[
\mathcal{P}_0: \text{Find } \mathbf{u} \in C^0 \left( (0, T), L^2(\Omega)^d \right) \cap L^2 \left( (0, T), \mathbb{V}(\mathbf{u}_b) \right) \text{ and } p \in L^2 \left( (0, T), \mathcal{Q} \right) \text{ such that}
\]

\[
\int_\Omega \rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) \cdot \mathbf{v} + \int_\Omega 2\mu \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) - \int_\Omega p \mathrm{div} \mathbf{v} + \sum_{i \in N} \int_{\Gamma_i} R (\mathbf{u} - \mathbf{u}^*) \cdot \mathbf{v} = \int_{\Gamma_N} (\mathbf{\sigma} \cdot \mathbf{n}) \cdot \mathbf{v}, \quad \forall \mathbf{v} \in \mathbb{V}(0),
\]

\[
\int_\Omega q \mathrm{div} \mathbf{u} = 0, \quad \forall q \in \mathcal{Q}.
\]

From now, the explicit dependence of \( \Omega \) and \( \Gamma_i, i \in N \) from \( t \) will be understood.

2.2.2. Extension and regularization. The variational formulation \( \mathcal{P}_0 \) involves integrals over the moving surfaces \( \Gamma_i, i \in N \). To avoid the explicit re-triangulation at each time of these surfaces, the integrals over \( \Gamma_i \) are transformed into integrals over the entire \( \Omega \). Remark that an integral over a closed surface \( S \) described as the zero level set of a given function \( \varphi \) can be written as an integral over \( \Omega \) with the help of the function \( \varphi \) and the Dirac measure \( \delta \):

\[
\int_S \eta \mathrm{d}s = \int_\Omega \tilde{\eta} |\nabla \varphi| \delta(\varphi) \mathrm{d}x,
\]

where \( \tilde{\eta} \) denotes an extension to \( \Omega \) of a function \( \eta \) defined on \( S \); see, for example, [54]. Nevertheless, handling Dirac measures is not easy, and a regularization procedure for \( \delta(\varphi) \) will be used. We introduce two additional sharp functions: the Heaviside function \( \mathcal{H}(\varphi) \) and the sign function \( \mathrm{sgn}(\varphi) \). A banded strip of width \( 2\varepsilon \) helps to perform the regularization procedure; and the functions \( \mathcal{H}, \delta, \) and
The terms corresponding to the resistive immersed surfaces in \( \mathcal{P}_0 \) are respectively replaced by \( \mathcal{H}_\varepsilon, \delta_\varepsilon, \) and \( \text{sgn} \). For all \( \beta \in \mathbb{R} \), the regularized functions write the following:

\[
\mathcal{H}_\varepsilon(\beta) = \begin{cases} 
0, & \text{when } \beta < -\varepsilon \\
\frac{1}{2} \left( 1 + \frac{\beta}{\varepsilon} + \frac{\sin(\pi \beta/\varepsilon)}{\pi} \right), & \text{when } |\beta| \leq \varepsilon, \\
1, & \text{otherwise}
\end{cases}
\]

2.2.3. Weak formulation. Let us consider \( H(\text{div}, \Omega) = \left\{ s \in (L^2(\Omega))^d : \text{div} s \in L^2(\Omega) \right\} \). We introduce the weighted multilinear forms:

\[
m(u, v; \phi) = \int_{\Omega} \phi u \cdot v \, dx, \quad \forall u, v \in (L^2(\Omega))^d \quad \text{and} \quad \phi \in L^\infty(\Omega),
\]

\[
a(u, v) = \int_{\Omega} 2 \mu D(u) : D(v) \, dx, \quad \forall u, v \in (H^1(\Omega))^d,
\]

\[
b(v, q) = -\int_{\Omega} q \text{div} v \, dx, \quad \forall q \in L^2(\Omega), \quad \forall v \in H(\text{div}, \Omega).
\]

The terms corresponding to the resistive immersed surfaces in \( \mathcal{P}_0 \) are regularized as follows:

\[
\int_{\partial \Omega} R \left( u - u^* \right) \cdot v \, ds \approx \int_{\partial \Omega} R \left| \nabla \varphi_i \right| \delta_\varepsilon(\varphi_i) \mathcal{H}_\varepsilon(1 - \psi_i) \left( u - u^* \right) \cdot v \, dx, \quad \forall i \in \mathcal{N}.
\]

We therefore consider the following regularized version of \( \mathcal{P}_0 \):

\[
\mathcal{P}_\varepsilon : \text{Given } \varphi_i \in L^2((0, T), W^{1,\infty}(\Omega)) \text{ and } \psi_i \in L^2((0, T), W^{1,\infty}(\Omega)), \text{ for all } i \in \mathcal{N} \text{ and } 0 < t < T \text{ find } u_\varepsilon \in \nabla(\mathcal{U}_b) \text{ and } p_\varepsilon \in \mathcal{Q} \text{ such that}
\]

\[
\int_{\Omega} \rho \left( \frac{\partial u_\varepsilon}{\partial t} + u_\varepsilon \cdot \nabla u_\varepsilon \right) \cdot v + a(u_\varepsilon, v) + b(v, p_\varepsilon) + \sum_{i \in \mathcal{N}} m(u_\varepsilon, v; R|\nabla \varphi_i| \delta_\varepsilon(\varphi_i) (1 - \mathcal{H}_\varepsilon(\psi_i)))
\]

\[
= \sum_{i \in \mathcal{N}} m(u^*, v; R|\nabla \varphi_i| \delta_\varepsilon(\varphi_i) (1 - \mathcal{H}_\varepsilon(\psi_i))) + \int_{\partial \Omega} (\sigma \cdot n) \cdot v,
\]

(2.4a)

\[
b(u_\varepsilon, q) = 0,
\]

(2.4b)

for all \( v \in \nabla(0) \) and \( q \in \mathcal{Q} \).

In what follows, the term \( \sum_{i \in \mathcal{N}} m(u^*, v; R|\nabla \varphi_i| \delta_\varepsilon(\varphi_i) (1 - \mathcal{H}_\varepsilon(\psi_i))) \) in (2.4a) will be simply denoted by \( \mathcal{F}_{u^*}(v) \). Note that \( \mathcal{F}_{u^*}(v) \) vanishes when the valve is either fully open or fully closed.

2.3. Initial and boundary conditions

To ensure that problem \( \mathcal{P}_\varepsilon(2.4a)-(2.4b) \) is well posed, suitable initial conditions are needed.

Let us consider \( \varphi_i,0 \in C^0(\Omega) \) and \( \psi_i,0 \in C^0(\Omega) \), for all \( i \in \mathcal{N} \). Initial conditions are readily applied as follows:

\[
u(t = 0) = u_0(\cdot), \quad \varphi_i(t = 0) = \varphi_{i,0}(\cdot) \quad \text{and} \quad \psi_i(t = 0) = \psi_{i,0}(\cdot), \quad \forall i \in \mathcal{N}.
\]

The inlet boundary condition. The inlet boundary \( \Gamma_{in} \) is located in correspondence with the lower level of the sinus of Valsalva at the output of the left ventricle. Inlet boundary conditions can be prescribed as a time-dependent mass flow rate, a velocity profile, or a pressure profile. Using a fully developed velocity profile is not appropriate as this would implicitly correspond to assuming a long straight vessel upstream the location of the inlet, which is not realistic because the leaflets are very close to the inlet boundary. Such a boundary condition can however be used for the outlet
boundary $\Gamma_{out}$. In such cases, magnetic resonance imaging techniques may provide measurements such as the velocity field and the flow rate profile. Regarding the flow rate boundary condition, different approaches such as the Lagrange multiplier technique helps to force this condition as a constraint for the fluid problem. The defective boundary condition can also be used yielding mainly to significantly reduce the high computational cost of the previous approach; see, for example, [55]. The choice of these boundary conditions will be discussed thoroughly in a further paper addressing simulations for patient-specific cases. In the present work, a pressure profile is prescribed. By so doing, we will consider $\Gamma_{in}$ as part of $\Gamma_N$, that is, $\Gamma_{in} \subset \Gamma_N$. As specified in [37], combining normal-traction and zero-tangential-slip boundary conditions, with the incompressibility constraint, along a flat boundary allows for the pointwise specification of the pressure on that boundary. Therefore, a periodic time-dependent left-ventricular pressure waveform $P_{lv}$ is used to drive the flow through the aortic valve. The normal component of the normal stresses helps to prescribe the pressure profile:

$$\sigma \cdot n = -P_{lv} n \quad \text{on} \quad (0, T) \times \Gamma_{in}.$$  

**The outlet boundary condition.** The outer boundary called $\Gamma_{out}$ is located on the top level of the ascending aorta, and it corresponds also to the lower part of the aortic arch. A prescribed physiological pressure waveform is also considered yielding $\Gamma_{out} \subset \Gamma_N$, and we use a three-element Windkessel model to obtain dynamic loading conditions for the valve model. The three-element Windkessel model is represented by an electrical circuit; see Figure 2. It accounts for the arterial wall compliance and the blood viscosity (i.e., the friction) through a capacitor $C$ and two resistances: a peripheral resistance $R_1$ and a systemic resistance $R_2$, as represented in Figure 2. By using a mass flow pulse as input, the Windkessel model responds with a physiological pressure waveform. The values of the resistances and the capacitance determine the pressure pulse waveform and its magnitude. We notice that the values of these parameter are usually obtained by *in vivo* measurements and by making reasonable assumptions; see, for example, [56, 57]. However, a tuning of these parameters is often needed, and it depends on the physical application [37]; see Section 4.1. The volumetric flow rate at the aortic terminal node $\Gamma_{out}$ is given by $Q(t) = \int_{\Gamma_{out}} \mathbf{u}(t, x) \cdot n \, ds$. By exploiting the electrical analogy, the Windkessel model leads to a differential relation between the aortic pressure $P_{ao}(t)$, the flow rate $Q(t)$, and the stored pressure $P_*(t)$ in the time domain. It reads as follows:

$$\frac{1}{R_1} (P_{ao}(t) - P_*(t)) = Q(t), \quad C \frac{dP_*(t)}{dt} - \frac{1}{R_2} P_*(t) = Q(t) \quad \text{and} \quad P_{ao}(0) = P_*(0) = 0. \quad (2.5)$$

Problem (2.5) is solved for the pressure $P_{ao}$. A physiological pressure load is then imposed on the valve throughout the cardiac cycle such that:

$$\sigma \cdot n = -P_{ao} n \quad \text{on} \quad (0, T) \times \Gamma_{out}.$$  

**The wall boundary condition.** For healthy persons, the aortic wall has a significant compliance, and it can be modeled as either smooth or rough in order to account for surface roughness. In the present work, we neglect the surface roughness, and we consider only a rigid wall. This is a
limitation; however, it is not too severe in case of elderly persons for which the aorta behaves almost like a rigid wall; see, for example, [58, 59]. Therefore, we prescribe a no-slip condition that simply consists in homogeneous Dirichlet condition for the velocity, meaning that frictional forces will create a boundary layer along the wall. Regarding the two ostia in the sinus of Valsalva, the no-slip boundary condition is also imposed, as we are not accounting for the coronary circulation in the present work. Hence, \( \Gamma_{\text{wall}} \) includes both the aortic wall and the coronary ostia.

To summarize, in our regularized problem \( \mathcal{P}_e \), we have taken \( \Gamma_D = \Gamma_{\text{wall}}, \Gamma_N = \Gamma_{\text{in}} \cup \Gamma_{\text{out}} \) and set \( u_0 = 0 \). Consequently, the last boundary integral in (2.4a) is replaced by the known term \( \mathcal{G} (P_{N}, P_{ao}) = - \int_{\Gamma_{\text{in}}} P_N n \cdot v - \int_{\Gamma_{\text{out}}} P_{ao} n \cdot v \).

3. NUMERICAL APPROXIMATION AND IMPLEMENTATION DETAILS

3.1. Time discretization

Let us divide \([0, T]\) into \(N\) sub-intervals \( [t^n, t^{n+1}], n = 0, \cdots, N-1 \) of constant step \( \Delta t \). Using uniform time steps is the most basic choice for simulations with laminar flows. However, the Reynolds number can achieve high values around the peak of systole, especially in the aortic stenosis cases. As this situation may trigger instabilities in the numerical computations, we reduce the time step when the velocity becomes larger than a critical threshold value, referred to as \( u_c \). In addition, \( \Delta t \) is reduced during the fast opening and closing of the valve. When the time step is adapted, we need to re-compute the aortic pressure by solving a second time the Windkessel model using the adapted time step instead of the uniform time step. The algorithm for the time evolution is reported in Algorithm 2.

**Algorithm 2** Time advancing strategy

1: set \( \Delta t^* = \Delta t \)
2: compute \( P_{ao} \) from (2.5)
3: test = moveValve(), see Alg. 1
4: if (test== 1) then
5: \( \Delta t = \Delta t^*/2 \)
6: else if (\( \xi = 1 \)&(\( u^{n-1}_e > u_c \)) then
7: \( \Delta t = \Delta t^*/10 \)
8: end if
9: compute \( \Gamma_{i,i \in N} \) from (2.1)
10: compute \( u^* \) from (3.1)
11: recompute \( P_{ao} \) (2.5) using \( \Delta t \)
12: compute \( (u^n_i, p^n_i) \) from \( \mathcal{P}_e \) (2.4a)–(2.4b)

For any \( n \geq 1 \), the unknowns \( u^n_i, p^n_i, \psi^n_i, \psi^n_i, P^n_i, \) and \( P^n_{ao} \) at time step \( t^n \) are computed. We choose the second-order backward differentiation formula for the momentum equation (2.4a) for its simple setup and its stability properties. The velocity time derivative term is then approximated by

\[
\frac{\partial u}{\partial t} (t^n) \approx \frac{3u^n - 4u^{n-1} + u^{n-2}}{2\Delta t}, \quad \text{with} \ u^n \approx u(t^n), \ n \geq 2,
\]

where we take the initial conditions \( u^0 = u(0) \) and \( u^1 \) computed using a second-order one-step explicit scheme. Concerning the time discretization of the Windkessel model, a second-order Runge–Kutta method is considered [60].

Finally, we focus on the time discretization of the valve velocity \( u^* \). For any \( i \in N \), the leaflet \( \Gamma_i \) is described using two level-set functions, and its velocity is given by the velocity of the primary level set \( \psi_i \). Every material point \( x(t) \in \Gamma_i \) shall verify the equation \( \psi_i(t, x(t)) = 0 \). The time derivation leads to the advection problem \( \partial_t \psi_i + u^* \cdot \nabla \psi_i = 0 \), where \( u^* = \partial_t x \) is the velocity of the front of \( \psi_i \). As \( \nabla \psi_i \) represents the normal to the surface of the leaflet \( \Gamma_i \), only the normal
component of the velocity has an effective contribution to the motion. Accordingly, we assume that the velocity of the leaflets writes $$u^* = |u^*| \nabla \psi_i / |\nabla \psi_i|$$. Finally, we obtain the following expression of the velocity of $$\Gamma_i$$:

$$u_i^{\ast,n+1} = -\frac{\psi_i^{n+1} - \psi_i^n}{\Delta t} \frac{\nabla \psi_i^{n+1}}{|\nabla \psi_i^{n+1}|}, \quad \forall i \in N. \quad (3.1)$$

3.2. Space discretization by finite elements

We consider a partition $$\mathcal{S}_h$$ of $$\Omega$$ consisting of geometrically conforming open simplicial elements $$K$$ (tetrahedra for $$d = 3$$, or triangles for $$d = 2$$), such that $$\Omega = \cup_{K \in \mathcal{S}_h}$$. The mesh size $$h = \max_{K \in \mathcal{S}_h} \text{diam}(K)$$ stands for the largest element diameter. We proceed with the space discretization of $$\mathcal{S}_\varepsilon$$. We introduce the following finite dimensional spaces:

$$Q_h = \{ q \in Q \cap C^0(\Omega) \, | \, q|_K \in P_1(K), \, \forall K \in \mathcal{S}_h \},$$

$$X_h = \{ u \in (C^0(\Omega))^2 \, | \, u_K \in (P_2(K))^d, \, \forall K \in \mathcal{S}_h \},$$

$$\forall_h(u_b) = X_h \cap \forall(u_b).$$

Thus, we look for $$u_{e,h}^{n,t}$$ and $$p_{e,h}^{n,t}$$ approximating $$u^n$$ and $$p^n$$, respectively, in time and space.

**Strategy I.** To fulfill the inf–sup condition, we consider the Taylor–Hood finite elements for the approximation of the velocity and the pressure [61, 62]. The discretization of $$\mathcal{S}_\varepsilon (2.4a)$$–(2.4b) reads as follows:

$$m \left( \frac{3u_{e,h}^n}{2\Delta t} + u_{e,h}^{n-1} \cdot \nabla u_{e,h}^n, v \right) + a(u_{e,h}^n, v)$$

$$+ \sum_{i \in N} m \left( u_{e,h}^n, v; R |\nabla \psi_i^{n-1}| \delta_{\varepsilon} \left( \psi_i^{n-1} \right) \left( 1 - \mathcal{C}_\varepsilon \left( \psi_i^{n-1} \right) \right) \right)$$

$$+ b(v, p_{e,h}^n) = m \left( \frac{4u_{e,h}^{n-1} - u_{e,h}^{n-2}}{2\Delta t}, v; \rho \right)$$

$$+ \mathcal{F}^{n-1}(v) + \mathcal{G}(P_v^n, P_{\omega}^n)$$

$$b(u_{e,h}^n, q) = 0. \quad (3.2a)$$

**Strategy II.** We consider a second-order extension of the characteristics method in this strategy. The second-order extrapolation of the velocity reads $$\tilde{u} = 2u^n - u^{n-1}$$. The first-order and second-order characteristics are given respectively by

$$X^n(x) = x - \Delta t \tilde{u}(x)$$

$$X^{n-1}(x) = x - 2\Delta t \tilde{u}(x).$$

The problem is approximated by the following second-order implicit Euler scheme:

$$\mathcal{S}_{\varepsilon,h}^{II} \, \text{Given} \, u_{e,h}^{n-2} \in \forall_h(u_b), u_{e,h}^{n-1} \in \forall_h(u_b), u_{e,h}^{n} \in \forall_h(u_b) \, \text{and} \, \left( \psi_{i,h}^{n-1}, \psi_{i,h}^{n-1} \right) \in Q_h \, \forall i \in N; \, \forall \, v \in \forall_h(0) \, \text{and} \, q \in \forall_h, \, \text{find} \, u_{e,h}^n \in \forall_h(u_b) \, \text{and} \, p_{e,h}^n \in \forall_h \, \text{such that}$$

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Figure 3. Idealized geometry of the aortic root and the leaflets. (a) Splines defining the external STL surface. (b–c) Lateral/bottom view of the wall and leaflets. (d) Mesh code M.

Table II. Dimensional parameters characterizing the leaflets.

<table>
<thead>
<tr>
<th>Aortic cusps (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter of left coronary cusp</td>
</tr>
<tr>
<td>Diameter of right coronary cusp</td>
</tr>
<tr>
<td>Diameter of noncoronary cusp</td>
</tr>
<tr>
<td>Height of left coronary cusp</td>
</tr>
<tr>
<td>Height of right coronary cusp</td>
</tr>
<tr>
<td>Height of noncoronary cusp</td>
</tr>
<tr>
<td>Commissure length</td>
</tr>
</tbody>
</table>

\[
m \left( \frac{3 \mathbf{u}_h^n}{2 \Delta t}, \mathbf{v} ; \rho \right) + a \left( \mathbf{u}_h^n, \mathbf{v} \right) + b \left( \mathbf{v}, p_h^n \right) \\
+ \sum_{i \in N} m \left( \mathbf{u}_i^n, \mathbf{v} ; R | \nabla \phi_i^n | \delta_i \left( \phi_i^n \right) \left( 1 - \mathcal{H}_i \left( \phi_i^n \right) \right) \right) \\
= m \left( \frac{4 \mathbf{u}_h^n \circ \mathbf{X}_h^n - \mathbf{u}_h^{n-2} \circ \mathbf{X}_h^{n-1}}{2 \Delta t}, \mathbf{v} ; \rho \right) + \mathcal{F}_h \mathbf{u}_h^{n-1} + \mathcal{G} \left( P_{wv}^n, P_{30}^n \right) \\
b \left( \mathbf{u}_h^n, q \right) = 0.
\]  

(3.3a)  

(3.3b)

3.3. Idealized geometry and leaflets

3.3.1. Realistic geometry for human aortic root. The geometry of the computational domain is made up of the aortic root and the ascending aorta, and it joins the aortic arch to the left ventricle, with several components including the aortic annulus, the Valsalva sinuses, the sino-tubular junction, and a portion of the two coronary ostia. We use a small set of anatomical parameters to build this geometry (Figure 3) taken from measurements reported in existing literature. In particular, our idealization follows the works of Labrosse et al. [49], Swanson et al. [63], Zhu et al. [64], and Buellesfeld et al. [65]. We adopt the same terminology introduced in [65], and we report the different characteristics of our geometry in Table II (right). Different unstructured meshes (fine (F), medium (M), and coarse (C)), generated by the free software GMSH\(^\ddagger\) [66] are used in our simulations; see Figure 3 and Table III. They are refined close to the coronaries and the domain of the leaflets in the open and closed positions. Several algorithms are used to build three-dimensional meshes, ensuring a good quality of the tetrahedral elements. For this purpose, the software MESHLAB\(^\S\) is used to locally smooth the initial STL triangulation of the external surface. A remeshing

\(^\ddagger\)GMSH, http://www.geuz.org/gmsh  
\(^\S\)MESHLAB, http://meshlab.sourceforge.net

Table III. Characteristics of the different meshes.

<table>
<thead>
<tr>
<th>Mesh code</th>
<th>Number of vertices</th>
<th>Number of tetrahedra</th>
<th>dof($P_1/P_1$)</th>
<th>dof($P_2/P_1$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>4,637</td>
<td>37,900</td>
<td>33,180</td>
<td>182,934</td>
</tr>
<tr>
<td>M</td>
<td>7,238</td>
<td>65,331</td>
<td>61,020</td>
<td>326,820</td>
</tr>
<tr>
<td>XM</td>
<td>13,327</td>
<td>119,154</td>
<td>110,964</td>
<td>594,882</td>
</tr>
<tr>
<td>F</td>
<td>20,145</td>
<td>174,210</td>
<td>158,708</td>
<td>858,956</td>
</tr>
</tbody>
</table>

technique is needed to prevent the numerical problems related to poor quality of distorted triangles. This is achieved by using the Frontal algorithm [66] that helps to remesh the original surface and to improve the geometrical aspect ratio of the mesh elements. The algorithm Netgen helps to optimize the quality of the tetrahedra. The unstructured tetrahedral meshes have moderate aspect ratio parameters, and the ratio between the minimal edge length and the maximal edge length in each tetrahedral element is kept always larger than 0.3.

3.3.2. Preprocessing: signed distance reconstruction of the primary level-set functions. The primary level-set functions $\varphi_i, i \in \mathcal{N}$, are represented with polynomial functions. Their gradients $\nabla \varphi_i$ can have very big or very small values near the leaflets. As either too large or too small gradients indicate steep or flat functions, respectively, their zero level sets are less accurately tracked, and the regularization parameter $\varepsilon$ will be chosen element-wise depending on the local values of $|\nabla \varphi_i|$. Inappropriate choice of $\varepsilon$ would lead to severe computational errors, and the fluid may be constrained to flow in areas bigger than a small surrounding of the leaflets’ surfaces. Indeed, the fluid can cross the leaflets in the zones where $|\nabla \varphi|$ is very small (i.e., we obtain a porous structure of leaflets), while the velocity vanishes in a large portion of the fluid domain around the leaflets if $|\nabla \varphi|$ becomes large.

To address this issue, we need to construct a distance function for both open and closed positions of the valve. More precisely, each primary level-set function $\varphi_i$ should approximate a signed distance especially in the vicinity of the valve. Therefore, we perform a preprocessing step in order to redress these functions as signed distance functions, and we solve an auxiliary problem called the redistancing problem, which helps to recover the distance property while keeping the initial position of the leaflets. An advection problem depending on a fictitious time variable $\tau$ is introduced, and we look for the stationary solution. Let $\hat{\varphi}_i, i \in \mathcal{N}$, be a known level-set function given by the algebraic representation that is far from a distance function. The redistancing problem reads as follows:

$$\frac{\partial \psi}{\partial \tau} (\tau, \cdot) + v \cdot \nabla \psi = \text{sgn} (\hat{\varphi}_i (t, \cdot)) + \lambda (\tau, \cdot) g (\psi) \quad \text{in } (0, +\infty) \times \Omega,$$

$$\psi (0, \cdot) = \hat{\varphi}_i (\cdot) \quad \text{in } \Omega.$$  

(3.4)

where the advection vector field reads $v = \text{sgn} (\hat{\varphi}_i) \nabla \psi / |\nabla \psi|$. The Lagrange multiplier $\lambda$ helps to enforce the constraint of preservation of the initial position of the leaflet. Further details are available in [67]. For both open and closed positions of each leaflet, we solve the redistancing problem (3.4) until convergence; the corresponding stationary solution $\psi(\infty, \cdot)$ approximates a signed distance function and will be used to provide the new description of the leaflet. For the time discretization of (3.4), we use the method of characteristics with a first-order backward Euler scheme.

4. NUMERICAL EXAMPLES

In what follows, we provide a set of numerical examples in both the two-dimensional and three-dimensional cases to test the model and the performance of the finite element method.

Software implementation. The presented method has been implemented using both LifeV¶ and Rheolef¶ environments. They are general purpose C++ libraries for scientific computing, with special emphasis on finite elements and parallel computation. The two libraries provide support for

distributed-memory parallelism via MPI\textsuperscript{**}. They rely upon the Trilinos\textsuperscript{††}, Boost\textsuperscript{‡‡}, Blas\textsuperscript{§§}, and UMFPACK\textsuperscript{¶¶} libraries for much of their functionalities. Rheolef bases on Scotch for distributed mesh partitioning\textsuperscript{|||}. Regarding LifeV, domain decomposition methods are used to design parallel preconditioners, based on the overlapping Schwarz preconditioner provided by the Ifpack package of Trilinos with full LU factorization for the subdomains, performed by the UMFPACK library. The preconditioned system (after linearization of the Navier–Stokes problem) is solved iteratively by a GMRES solver provided by the AztecOO package of Trilinos. The computational results are displayed graphically using the software PARAVIEW\textsuperscript{***}, while the plots are generated using the software GNUPLOT\textsuperscript{†††}.

4.1. Cylinder test case: pressure profile and setting of parameters

We have set this first test to better highlight several features of our model. This numerical experiment helps to tune different numerical parameters and to test, in particular, the three-element Windkessel model without using a realistic aortic valve. The setup is as follows: the fluid domain is a cylinder with radius $R = 1$ cm and height $H = 5$ cm. The lumped model that serves to compute the opening angle of the valve is not activated in this test case, which amounts to suppose that the aortic valve switches between two positions. The closed valve $\Gamma$ has a circular shape, and it is located in the middle of the cylinder (Figure 4 (left)), while the open valve represents an empty set, that is, the resistance $R$ becomes equal to zero. The valve is closed if the downstream pressure is bigger than the upstream pressure, and it is open otherwise. The fluid domain is discretized using 73,642 tetrahedral elements using GMSH, and we choose a time step size equal to $5 \times 10^{-3}$ s. The model is driven by a periodic left ventricular pressure imposed at the upstream boundary (Figure 5(a)), and we run the simulation over multiple heart beats until a nearly periodic pressure profile is reached at the downstream boundary $\Gamma_{\text{out}}$. The pressure profile at $\Gamma_{\text{out}}$ is provided by the system (2.5). The parameters of the Windkessel model are numerically tuned, and they are given by $R_1 = 160$ dyn s cm$^{-5}$, $R_2 = 2420$ dyn s cm$^{-5}$, and $C = 9.5 \times 10^{-4}$ cm$^5$ dyn$^{-1}$. During the early heartbeats of the simulation, we observe that the pressure is not periodic, before it approaches afterwards a periodic steady state; see Figure 5. Consequently, the flow rate computed on $\Gamma_{\text{out}}$ and the valve state $\xi$ become both periodic. We set the resistance value equal to $10^6$. Some snapshots of the pressure and the velocity profiles are reported in Figures 4 and 6. They correspond to the results obtained during the fifth heartbeat, respectively, for $t = 3.36$ s (diastolic phase) and $t = 3.49$ s (systolic phase); see Figure 6 (left). When the valve is open ($t = 3.49$ s), the gradient of the pressure is clearly visible, and the maximum of the velocity approximates 168 cm s$^{-1}$. However, during the diastolic phase, the valve is closed and the velocity becomes small. Despite the use of an Eulerian formulation, a sharp pressure

Figure 4. Left: The computational domain with the circular valve $\Gamma$ in the closed position. Middle: The velocity field when the valve is closed. Right: The velocity field when the valve is open.
Figure 5. Numerical results from multibeat computations of the two-position circular valve in the cylindrical geometry. (a) The prescribed upstream pressure and the computed downstream pressure. (b) The flow rate computed on the outlet.

Figure 6. Left: The different times considered. Middle: The pressure jump when the valve is closed (time = 3.36 s). Right: The pressure profile when the valve is open (time = 3.49 s).

Figure 7. Left: Dependence on the regularization parameter $\varepsilon$. Middle: Dependence on the time step size $\Delta t$. Right: Dependence on the resistance parameter $R$.

A gradient is observed across the valve as depicted in Figure 6 (middle), where the pressure gradient is equal to $10^4$ dyn cm$^{-2}$. Indeed, the mesh does not fit the valve surface, and the pressure jump is consequently regularized in a banded region having a width of $2\varepsilon$ in the vicinity of the zero level set; see Section 2.2.2.

The problem contains three numerical parameters, the time step $\Delta t$, the regularization parameter $\varepsilon$, and the resistance $R$, which need to be properly tuned. Let us consider fixed pressure values on the inlet and the outlet, which correspond to the values of the pressure in the previous simulation at the time $t = 3.36$ s. Hence, the valve is closed, and we compute the flow rate downstream the valve. In the first experiment, we set $R = 10^4$ and $\Delta t = 5 \times 10^{-3}$, and we compute the flow rate for several values of the band width $2\varepsilon$. Figure 7 (left) shows that the band width should be small enough to penalize the flow across the valve, but not too much to prevent the blood circulation in areas far from the valve. In Figure 7 (left), the value $\varepsilon = 0.3$ approximately corresponds to three...
mesh elements. Snapshots of the pressure jump across the valve are provided in Figure 8 for three different values of the regularization parameter $\varepsilon$.

In the second experiment, we set $R = 10^4$ and $2\varepsilon = 3h$, where $h$ stands for the mean value of the mesh size, and we vary the time step size. Based on the results depicted in Figure 7 (middle), we decide to take $\Delta t \leq 5 \times 10^{-3}$ s. We recall that this value will be adapted afterwards with respect to the valve state and the value of the velocity module.

Finally, we investigate the choice of the resistance using the fixed values $\Delta t = 5 \times 10^{-3}$ and $2\varepsilon = 3h$. The resistance has an important impact on the flow rate across the valve, and it has to be chosen quite large to penalize the velocity on the valve. Based on the results in Figure 7 (left), we choose a resistance around $10^6$. Larger values of $R$ would harm the conditioning of the resulting linear system.

4.2. Numerical computation of the level-set velocity $u^*$

Before testing the entire valve model, we introduce in this paragraph an elementary test to assess the accuracy of the numerical computation of the valve velocity $u^*$ (3.1). With this aim, we use a simple square domain $[0, 1]^2$ in the two-dimensional case. We consider a rigid body rotation of a circle in a constant rotating velocity field $u(x, y) = (1/2 - y, x - 1/2)^T$. A level-set description was used. The initial circle of radius $1/5$ is centered in $(1/2, 7/10)$ and performs a full revolution in one period equal to $2\pi$ s. In Figure 9, the original velocity $u$ (left), the normal velocity $u \cdot \nabla \varphi$ (middle), and the computed velocity $u_h^*$ (right) are shown.

Figure 8. Snapshots showing the pressure jump across the closed valve at time $t = 3.36$ s for several values of $\varepsilon$. Left: $\varepsilon = 0.15$. Middle: $\varepsilon = 0.35$. Right: $\varepsilon = 0.6$.

Figure 9. Numerical computation of the level-set velocity. Left: Rotating velocity field $u$. Middle: Normal component of the velocity field (i.e., exact solution) $u^*$. Right: Computed velocity $u_h^*$. 

(middle), and the computed velocity \( u_h^* \) are represented at two different instants. The snapshots show the qualitative agreement between the effective velocity, that is, the normal velocity, and the computed solution \( u_h^* \). We use the \( P_2 \) finite element approximation for the velocity, and we compute the error between the computed velocity \( u_h^* \) and the exact value \( \pi_h u^* \), where \( \pi_h \) denotes the Lagrange interpolation in the velocity space. The errors are evaluated in the vicinity of the zero level set \( \Gamma \) using the \( L^2(\Gamma) \) and \( L^\infty(\Gamma) \) norms. The rates of convergence, that is, the slope on the logarithmic scale, obtained numerically are equal to 2 and 1, respectively; see Figure 10.

4.3. Two-dimensional test case

4.3.1. Simulations under physiological pressure loads. Although the flow through the aortic valve is inherently three-dimensional, it is useful to study the flow pattern and to test the model in a simple two-dimensional case. Flow through an idealized aortic valve was therefore investigated under physiological pulsatile flow conditions. The geometry and the dimensions of the aortic root and the sinuses of Valsalva are based on the data from [68], and they are obtained using data of human aortic valve [69]. The width of the vessel is equal to 2.4 cm, while the length of the leaflets and the total length of the vessel are equal to 1.2 cm and 6 cm, respectively. The geometry was designed using GMSH, and the closed position of the aortic valve is represented by two symmetric circular leaflets attached to the aortic root. The computational domain was discretized by 24,540 triangular elements. For any mesh element, the geometrical aspect parameter representing the ratio between the minimal and the maximal edge lengths is bigger than 0.61. The flow domain is depicted in Figure 13(a).

The viscosity of blood is assumed equal to 0.035 g cm\(^{-1}\) s\(^{-1}\), and the flow through the aortic valve was simulated by imposing a physiological pressure at the ventricular and aortic sides. The prescribed pressure profiles at the inflow boundary \( \Gamma_{in} \) and the outlet \( \Gamma_{out} \) are shown in Figure 11 (left). We would like to remark that we did not solve the reduced Windkessel model in this two-dimensional case. The flow was simulated over multiple cardiac cycles using a constant time step size of \( 2 \times 10^{-3} \) s. Numerical results show that the instantaneous flow field downstream of the valve exhibits a very rich dynamics. The flow rate computed on \( \Gamma_{out} \) during the two first heart beats is plotted in Figure 11 (right). The peak systole approximately corresponds to the time 0.435 s, where
Figure 12. Snapshots of the velocity and pressure profiles in the two-dimensional case.

the valve is completely open and the jet is fully developed, and the maximum velocity is around $267 \text{ cm s}^{-1}$. The blood is strongly ejected through the aortic valve, and a recirculating flow zone appears near the tips of the leaflets; see the flow pattern at time 0.435 s in Figure 12. As the simulated valve is a healthy one, we can observe that the vortices that formed have similar sizes on the top of the two leaflets. Besides, a transvalvular pressure jump is also observed, and the pressure in the middle part of the vessel is bigger than the pressure in the two sinuses; see the pressure plots at time 0.275 s and time 0.305 s in Figure 12.

The valve remains in the open position as long as the pressure is bigger in the left ventricle, and the closure of the valve is subjected to the flow reversal in the opposite direction of the valve opening. The pressure gradient between the two sides of the valve, that is, in the aorta and the left ventricle, decreases during the deceleration phase until an adverse pressure gradient occurs, and it contributes to the valve closure. The two leaflets start to close simultaneously. The fluid in the aorta moves backward into the sinuses of Valsalva, and it is also fed by the two symmetrical vortices observed in the upper regions to the two leaflets. Therefore, a more perturbed flow dynamics occurs; see the velocity at time = 0.49 s and time = 0.54 s in Figure 12. As the aortic valve closes quickly (Figure 11 (middle)), the closure induces large vortices that are essentially developed in the sinuses of Valsalva; see for instance the flow pattern at time 0.77 s and time 0.995 s in Figure 12. A sharp pressure jump is observed across the distal side and the proximal side of the leaflets when the valve is completely closed; see the pressure plots at time 0.48 s and time 1 s in Figure 12.

4.3.2. Steady-state simulations. This test concerns the study of the flow profile in a simple case where the flow is laminar and not vortical. Therefore, we prescribe a constant pressure gradient across the aortic valve: the left ventricular pressure is equal to $150 \text{ dyn cm}^{-2}$, while the aortic pressure is equal to 100 dyn cm$^{-2}$. We impose a fixed value of the valve state parameter $\xi = 0.9$, which corresponds to an intermediate opening position of the leaflets, and we use the physiological blood viscosity $\mu = 0.035 \text{ g cm}^{-1} \text{s}^{-1}$ at $37^\circ \text{C}$. The problem is solved for several mesh sizes until we reach the steady state characterized by a residual smaller than $10^{-6}$; see Figure 13 (b). We plot, in particular, the cut of the velocity components $u = (u_1, u_2)$ along the horizontal line passing through the point (3.1, 4.7), and the cut of the $y$-component of the velocity along the vertical median line.
Figure 13. Steady-state simulations in the two-dimensional case. (a) Schematic view of the mesh domain. (b) Evolution of the residual in the logarithmic scale.

Figure 14. Velocity profile along lines passing through the point (3.1, 4.7) for different mesh sizes. Left: The $x$-component along the horizontal line. Middle: The $y$-component along the horizontal line. Right: The $y$-component along the vertical line.

Figure 14 depicts the velocity components for the particular line scans, where the mesh sizes are respectively given by $h_1 = 0.065$ mm, $h_2 = 0.1$ mm, $h_3 = 0.2$ mm, and $h_4 = 1.2$ mm. In particular, we observe that the velocity component $u_1(x, 4.7)$ has two local maxima and two local minima. The velocity component $u_2(x, 4.7)$ shows clearly the flow laminarity, while the two symmetric minima correspond to the two small vortices that are created near the tips of the leaflets. However, the velocity component $u_2(3.1, y)$ along the vertical median shows that only one maximum exists, and it corresponds approximately to the top of the sinuses of Valsalva. Although the solution corresponding to the mesh size $h_4$ deviates somewhat, the velocity profiles seem to converge toward the same profile when the mesh size becomes very small.

4.4. Qualitative comparison between the two solution strategies

The purpose of this test case is to perform a qualitative comparison between the numerical results obtained with two different strategies; see Section 3.1. Numerical computations are performed with LifeV using strategy I and with Rheolef using strategy II. For the sake of comparison, the two experiments are carried out using almost the same set of parameters, and the computational domain is approximated with the same mesh having 37,900 tetrahedral elements. The fluid density and viscosity are set respectively to $\rho = 1$ g cm$^{-3}$ and $\mu = 0.04$ g cm$^{-1}$ s$^{-1}$, and the fluid is assumed to be initially at rest. We emphasize that the aim of this test case is not to provide physiologically relevant numerical results. The model is driven by a periodic pressure waveform that is ten times smaller than the physiological pressure; see Figure 15 (left). At the outlet, the pressure is computed by solving the three-element Windkessel model using the same parameters tuned for the cylinder.
test case; see Section 4.1. We run the simulations until a stationary pressure waveform is observed on the outlet. This state corresponds to the third heartbeat for both strategies.

The computational results of the fourth and fifth heartbeats are reported in Figure 15 for the pressures and flow rates on $\Gamma_{out}$. We observe that the flow rates have similar profiles except around the peak of the systolic phase. The peak flow rate is approximately $42.7 \text{ cm}^3 \text{s}^{-1}$ for strategy I, a little lower than the flow rate $Q = 45.68 \text{ cm}^3 \text{s}^{-1}$ obtained using strategy II. Remark that both the opening and the closure of the valve happen approximately at the same times in the two experiments. The computed loading pressure profiles on the ascending aorta depend obviously on the flow rate, and we can see that the maximum systolic pressure has been reached faster in strategy II, following, thus, the flow rate profile. However, the diastolic pressure has somehow similar profiles, especially during the fourth heartbeat. Some snapshots showing the velocity profile around the systolic peak are reported in Figure 16.

Although the two sets of results are almost similar, we will use strategy II involving the characteristic method to simulate our physiologically relevant test cases as this method features slightly better stability properties when the Reynolds number becomes higher.

4.5. Numerical results under physiological pressure loading

Let us focus now on the physiologically relevant computations. A realistic pressure profile is imposed on the inner boundary (Figure 17(a)), and we compute the pressure on the downstream boundary by solving the three-element Windkessel model. The motion of the valve is provided by solving the reduced model discussed in Section 2.1.2.

A realistic and nearly periodic pressure waveform on $\Gamma_{out}$ was obtained after simulating approximately six heartbeats. Figure 17(c) shows that the peak flow rate is about $280 \text{ cm}^3 \text{s}^{-1}$, higher than the peak flow rate reported in [5] (about $180 \text{ cm}^3 \text{s}^{-1}$) and lower than the computed flow rate in [37] (which is about $420 \text{ cm}^3 \text{s}^{-1}$). The pressure waveform has a smoother profile compared with the pressure reported in [37], and we speculate that this behavior is mainly related to the unphysiological rigidity of the aortic root, and to the stroke volume observed in [37], which is related to the nonperfect closure of the valves. Numerical computations show that the pulse pressure, which is computed as the difference between the systolic and diastolic pressures, is about $54,086 \text{ dyn cm}^{-2}$. Indeed, the pulse pressure is commonly used to describe the force generated by the heart each time it contracts, and it is strongly correlated to the quantity of blood ejected during systole from the left ventricle. The obtained value is in good agreement with the corresponding experimentally obtained value of $40 \text{ mmHg}$ [70].

The valve opening and closing phases are very fast, and they are observed only if the time step is small enough. We report in Figures 18 and 19 some snapshots of the shape of the leaflets during these two phases. A more detailed study of these two phases will be carried out in a separate work.

Figure 20 provides some snapshots of the velocity and pressure profiles when the valve is either fully open or fully closed, for four different time instants. Computational results clearly show that...
the velocity is almost equal to zero in the vicinity of the valve during the diastolic phase, while a sharp jump of the pressure is observed across the leaflets. The ejection phase lasts about 0.26 s, and the velocity increases rapidly as soon as the valve opens. The velocity is characterized by a peak amplitude of about 180 cm s$^{-1}$.

4.6. Numerical simulation of a pathological valve in stenotic and regurgitant conditions

In this section, we consider an aortic valve having only one pathological leaflet denoted by $\Gamma_1$, while the remaining leaflets $\Gamma_2$ and $\Gamma_3$ are in a healthy state. The three leaflets are not symmetric to each
Figure 17. Numerical results from a multibeat simulations of the healthy aortic valve. Top row: Prescribed left ventricular pressure and computed aortic pressure. Middle row: Valve state parameter $\xi$. Bottom row: Flow rate computed on the outlet.

Figure 18. Shapes of the aortic valve during the opening phase, respectively, for $\theta = 5^\circ$, $23^\circ$, $64^\circ$, $74^\circ$, and $75^\circ$ at times $t = 0.117, 0.122, 0.126, 0.13$, and $0.152$ s (from left to right). Top: Velocity profile. Bottom: Shapes.

other anymore, and the stenotic valve has the pathological leaflet $\Gamma_1$ that opens only up to $26^\circ$, while $\Gamma_2$ and $\Gamma_3$ fully open up to $75^\circ$; see Figures 21 (left) and 22 (middle). The aortic valve stenosis is essentially characterized by the aortic pressure being much smaller than the left ventricular pressure during left ventricular ejection. In this test case, we keep using the same profile of the left ventricular pressure as the healthy case, where the peak pressure is about $1.5 \times 10^5$ dyn cm$^{-3}$. We compute the aortic pressure by means of the three-element Windkessel model. By comparison with the healthy case, computations show that the peak systolic pressure is reduced from $1.23 \times 10^5$ dyn cm$^{-3}$ to $9.5 \times 10^4$ dyn cm$^{-3}$; see Figure 22 (left). Therefore, the pressure gradient across the leaflets becomes much higher than the the healthy case, and even drastically higher in the case of severe stenosis. In fact, the stenotic valve induces high pressure gradient across the leaflets mainly for two reasons: the narrowing in the open position of the valve (i.e., an increased resistance) and the turbulence that occurs in the ascending aorta. In the present
Figure 19. Shapes of the aortic valve during the closing phase, respectively, for $\theta = 75^\circ$, $75^\circ$, $58^\circ$, $16^\circ$, and $5^\circ$ at times $t = 0.31, 0.342, 0.36, 0.362$, and $0.367$ s (from left to right). Top: Velocity profile. Bottom: Shapes.

Figure 20. Numerical simulations of the healthy aortic valve for four different time instants. Top: Velocity distribution. Bottom: Pressure profile.
test, we do not consider a case of severe stenosis where the vortices are much more important, and the fluid model shall include in this case the modeling of the small eddy scale and the underlying vortices.

Figure 22 (left) shows how the aortic stenosis may affect the aortic pressure during the cardiac cycle. The shaded area between the left ventricular pressure and the aortic pressure during the systole represents the elevated pressure gradient characterizing the aortic stenosis. As the minimal angle is assumed unchanged in the stenotic valve, the larger gradient only occurs during the ejection of blood across the valve. In clinical practice, the measurements of the previous quantity using the catheterization technique provide a quantitative assessment of the severity of the stenosis. Numerical results reveal that the jet profile across the valve is more important in the stenotic case. The peak flow rate is about 500 cm$^3$/s, and it is remarkably bigger than the maximal flow rate in the healthy case.

Moreover, we can observe a stroke volume during the closing process, as the valve movement is not anymore synchronized as shown in Figure 22 (right). Indeed, the leaflet $\Gamma_1$ closes more rapidly than $\Gamma_2$ and $\Gamma_3$; see Figure 22 (middle). Finally, we provide some snapshots showing the velocity distribution in the stenotic case; see Figure 23. We remark a modification in the flow profile compared with the healthy case; see Figure 20 (top row). The jet becomes closer to the turbulent regime, and the increase in the peak velocity is remarkable.

The final test is performed to illustrate the capability of the present model to simulate other valve pathologies such as a regurgitant, or leaky, valve. The leaflets $\Gamma_2$ and $\Gamma_3$ open and close completely, whereas the diseased leaflet $\Gamma_1$ opens completely, but it closes only up to 40°. Figure 24 shows snapshots of the velocity field for successive time steps during the diastolic phase. As the leaflet $\Gamma_1$ does not close completely, the blood leaks backward across the aortic valve as depicted in Figure 25, while the maximal flow rate during the systolic phase is smaller than the maximal flow rate in the healthy case. The mean backward flow, or regurgitant flow, is approximately 25 cm$^3$/s.
Figure 23. Velocity distribution during the systolic phase for the stenotic pathological aortic valve.

Figure 24. Velocity distribution during the diastolic phase for the regurgitant pathological aortic valve.

Figure 25. Numerical simulation of the regurgitant aortic valve. Left: Pressure profile. Middle: Opening angle. Right: Flow rate computed on $\Gamma_{out}$.

4.7. Scalability and parallel performance

As the scalability, also referred to as scaling efficiency, of the code represents an important aspect for our application, this section deals with the assessment of the parallel performances of our numerical solver. The scalability measures the ability of a parallel system to maintain a constant efficiency, and it is evaluated by computing, for instance, the speed up obtained when doubling the number of processors involved. The speed up compares the time required on a parallel computer to the time required on a sequential computer. The load balancing aims at providing to each processor, that is, CPU, the same proportion of work to carry out, so that no processor remains idle while waiting for the others to accomplish their task.
Figure 26. Left: Parallel performances of the numerical method. Timing for the assembly and the computation of the solution of the linear system for different meshes III and numbers of CPUs.

On the one hand, the strong scalability explores how fast the efficiency, or the solution time, changes by increasing the number of processors used to solve a problem having a fixed size. In the case of perfect strong scalability, the program scales linearly, and a two times faster execution is achieved if we double the number of processors. We represent the perfect strong scalability by a slope equal to one in Figure 26. We notice that the greater the slope is, the more strongly scalable is the method. This aspect is important if we simply want to speed up the numerical computations. On the other hand, the weak scalability investigates how the solution time differs from the number of CPUs for a fixed problem size per processor. The weak scalability is interesting if we want to use a finer mesh and increase the number of processors such that we keep a similar computational time. The perfect weak scalability corresponds to a perfect overlap of the curves corresponding to the use of different meshes. The closer the curves, the more weakly scalable is the method.

In Figure 26, we plot the timings of the computations with respect to the degrees of freedom per CPU, for the different meshes introduced in Table III. We can clearly observe a good strong scaling because the slopes of the different curves are quite close to the optimal linear slope. Moreover, the weak scalability of the code is well observed because the various timings’ curves are almost superimposed.

5. CONCLUSIONS

In this paper, we have presented a new model for heart valves using an Eulerian finite element formulation that handles the opening and the closure of valves without difficulties. An approach modeling the valve as a discrete set of two resistive immersed surfaces originally proposed in [5] was suitably generalized. A level-set approach helps to model and follow in an easier way the movement of the leaflets, allowing the use of a unified mesh for valves and fluid. Our numerical scheme is based on finite elements, and, over multiple cardiac cycles, we have reported numerical simulations in both two-dimensional and three-dimensional cases using physiological pressure profiles. Results address in detail the efficiency and the relevance of the mathematical model in physiological conditions. A qualitative comparison between the computational results obtained by two independent finite element strategies shows that it is possible to obtain a good agreement, and the computed quantities were in the same range. Although the present model does not include any specification on the elastic properties of the valve, it allows for the accurate study of the hemodynamics, and it allows substantial computational savings compared with full FSI simulations. Our model was also able to simulate the flow dynamics in the case of pathological heart valves, in particular stenotic and regurgitant aortic valves.

Nevertheless, the present model features several limitations common to other numerical models. Some extensions of this model are currently being explored, in particular regarding mitral and tricuspid valves that require developing a more complex valve model than the aortic and pulmonary valves (to include the chordae tendineae and the papillary muscles). As the elastic properties of these valve are clinically relevant, specification on the anisotropic elastic properties of the leaflets needs...
to be incorporated in the present model. Modeling of the complete set of valves in the full heart and studying of the flow pattern inside the four cardiac chambers are still too challenging and represent the final goal of our current developments.

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