Deformable contour models

Slide sources: original from Kristen Grauman; updated by Vittorio Ferrari
The grouping problem

- Goal: move from array of pixel values to a collection of regions, objects, and shapes
- Relying on shape priors
  - Specific assumptions, e.g. Hough transform
  - More generic priors
Given a model of interest, we can overcome some of the missing and noisy edges using **fitting** techniques. With voting methods like the **Hough transform**, detected points vote on possible model parameters.
Active contour models: Snakes

Interactive approach
Given: initial contour (model) near desired object

(Single frame)

Fig: Y. Boykov

Slide sources: original from Kristen Grauman
Snakes

Interactive approach
Given: initial contour (model) near desired object
Goal: evolve the contour to fit exact object boundary

(Single frame)

[Snakes: Active contour models, Kass, Witkin, & Terzopoulos, ICCV1987]
Snakes

Initialize near contour of interest
Iteratively refine: elastic band is adjusted so as to
• be near image positions with high gradients, and
• satisfy shape “preferences” or contour priors
• requires initialization near object of interest
• one optimization “pass” to fit a single contour

Fig: Y. Boykov
Deformable contours: intuition

Image from http://www.healthline.com/blogs/exercise_fitness/uploaded_images/HandBand2-795868.JPG

Figures from Shapiro & Stockman
Contour adjustment

How is the current contour adjusted to find the new contour at each iteration?

• Define energy function that says how good a configuration is
• Seek next configuration that reduces energy

initial          intermediate          final
Snakes energy function

The total energy of the current snake is:

\[ E_{total} = E_{internal} + E_{external} \]

**Internal** energy: encourage prior shape preferences: e.g. smoothness, elasticity, known shape prior.

**External** energy (image energy): encourage contour to fit interesting image structures, e.g. edges.

A **good** fit between the current snake and the target shape in the image will yield a **low** energy value.
Parametric curve representation

Continuous case

\[ \nu(s) = (x(s), y(s)) \quad 0 \leq s \leq 1 \]
External (image) energy: intuition

• Measure how well the curve matches the image data
• Attract the curve toward interesting image features
  – Edges, lines, etc.
How do edges affect “snap” of rubber band?

Think of external energy from image as gravitational pull towards areas of high contrast.

Magnitude of gradient

\[ G_x(I)^2 + G_y(I)^2 \]

- (Magnitude of gradient)

\[ -\left( G_x(I)^2 + G_y(I)^2 \right) \]
External (image) energy

- Image \( I(x,y) \)
- Directional derivatives
  \[ G_x(x,y) \quad G_y(x,y) \]
- External energy at a point \( v(s) \) on the curve is
  \[ E_{\text{external}}(v(s)) = -\left( |G_x(v(s))|^2 + |G_y(v(s))|^2 \right) \]
- External energy for the whole curve:
  \[ E_{\text{external}} = \int_0^1 E_{\text{external}}(v(s)) \, ds \]
A priori, we want to favor smooth shapes, contours with low curvature, contours similar to a known shape, etc. to balance what is actually observed in the gradient image.
For a continuous curve, a common internal energy term is the “deformation energy”. At some point \( v(s) \) on the curve, this is:

\[
E_{\text{internal}}(v(s)) = \alpha \left| \frac{d v}{d s} \right|^2 + \beta \left| \frac{d^2 v}{d^2 s} \right|^2
\]

The more the curve stretches and bends \( \rightarrow \) the larger this energy value is.

The weights \( \alpha \) and \( \beta \) dictate how much influence each component has.

Internal energy for whole curve:

\[
E_{\text{internal}} = \int_{0}^{1} E_{\text{internal}}(v(s)) \, ds
\]
Dealing with missing data

The smoothness constraints imposed by the internal energy helps dealing with missing data:

[Figure from Kass et al. 1987]
Total energy

continuous form

\[ E_{total} = E_{internal} + E_{external} \]

\[ E_{internal} = \int_{0}^{1} E_{internal}(\nu(s)) \, ds \quad \text{deformation energy} \]

\[ E_{external} = \int_{0}^{1} E_{external}(\nu(s)) \, ds \quad \text{total edge strength under curve} \]
Parametric curve representation

- Curve discretization
- Represent the curve with a set of $n$ points

$$\nu_i = (x_i, y_i) \quad i = \ldots \ldots n - 1$$
Discrete energy function: external term

The curve is represented by \( n \) points

\[
E_{\text{external}} = - \sum_{i=0}^{n-1} \left| G_x(x_i, y_i) \right|^2 + \left| G_y(x_i, y_i) \right|^2
\]

\( G_x(x, y) \) and \( G_y(x, y) \) are discrete image derivatives.
Discrete energy function: internal term

The curve is represented by $n$ points

$$v_i = (x_i, y_i) \quad i = 0 \ldots n-1$$

$$\frac{d\nu}{ds} \approx v_{i+1} - v_i$$

$$\frac{d^2\nu}{ds^2} \approx (v_{i+1} - v_i) - (v_i - v_{i-1}) = v_{i+1} - 2v_i + v_{i-1}$$

$$E_{\text{internal}} = \sum_{i=0}^{n-1} \alpha \|v_{i+1} - v_i\|^2 + \beta \|v_{i+1} - 2v_i + v_{i-1}\|^2$$

Elasticity, Tension

Stiffness, Curvature
Function of the weights

\[ \sum_{i=0}^{n-1} \alpha \| \nu_{i+1} - \nu_i \|^2 + \beta \| \nu_{i+1} - 2\nu_i + \nu_{i-1} \|^2 \]

\( \alpha, \beta \) weights control the penalty for deformation
( \( \alpha \) for stretching and \( \beta \) for bending)

Fig from Y. Boykov
If object is some smooth variation on a known shape, we can add a term to penalize deviation from that shape:

$$\delta \sum_{i=0}^{n-1} (v_i - \hat{v}_i)^2$$

where \( \{ \hat{v}_i \} \) are the points of the known shape.
Several methods proposed to fit snakes, including methods based on:

- Partial Differential Equations (PDEs)
- Greedy search
- Dynamic programming (for 2d snakes)
Energy minimization through PDEs

Energy to minimize (continuous case)

\[
\frac{1}{2} \int_0^1 \left( \alpha(s) \left\| \frac{\partial v}{\partial s} \right\|^2 + \beta(s) \left\| \frac{\partial^2 v}{\partial s^2} \right\|^2 \right) ds - \int_0^1 P(v) ds
\]

- deformation energy
- image energy (gradients)

Variational calculus $\rightarrow$ Euler-Lagrange differential equation

\[
- \frac{\partial}{\partial s} \left( \alpha(s) \frac{\partial v}{\partial s} \right) + \frac{\partial^2}{\partial s^2} \left( \beta(s) \frac{\partial^2 v}{\partial s^2} \right) = -\nabla P(v)
\]
Energy minimization through PDEs

with weights uniform over the snake

\[-\alpha \frac{\partial^2 x}{\partial s^2} + \beta \frac{\partial^4 x}{\partial s^4} = -\frac{\partial P}{\partial x}\]

\[-\alpha \frac{\partial^2 y}{\partial s^2} + \beta \frac{\partial^4 y}{\partial s^4} = -\frac{\partial P}{\partial y}\]

in short notation

\[-\alpha v_{ss} + \beta v_{ssss} = -P_v\]

find solution in iterative fashion (using image gradients at position in previous iteration $t-1$)

\[-\alpha v_{ss}^t + \beta v_{ssss}^t = -P_v \bigg|_{v=v^{t-1}}\]
Energy minimization through PDEs

Implementation:
discretize curve into \( n \) points \( V_i \)

use a finite difference approximation of derivatives

\[
\begin{align*}
v_{ss}^t(s_i) &\approx v_{i-1}^t - 2v_i^t + v_{i+1}^t \\
v_{ssss}^t(s_i) &\approx v_{i-2}^t - 4v_{i-1}^t + 6v_i^t - 4v_{i+1}^t + v_{i+2}^t
\end{align*}
\]

Substitute in iteration scheme above \( \rightarrow \) system of linear equations

\[
\mathbf{K}v^t = -P_v \bigg|_{v=v^{t-1}}
\]

Stiffness matrix \( \mathbf{K} \):
- made of coeffs \( \alpha, \beta \)
- penta-diagonal

Solve by inverting \( \mathbf{K} \)

\[
v^t = \mathbf{K}^{-1}P_v \bigg|_{v=v^{t-1}}
\]
Inversion of the stiffness matrix

\[ v^t = K^{-1} P_v \mid_{v=v^{t-1}} \]

Problem: \( K \) is singular

Unique solution needs boundary conditions

Closed contour: could be given implicitly (periodicity)

\[ v(0) = v(1), \quad v'(0) = v'(1), \quad v''(0) = v''(1), \quad v'''(0) = v'''(1) \]

Discretization

\[ v_n \Rightarrow v_0, \quad v_{n+1} \Rightarrow v_1, \quad v_{-1} \Rightarrow v_{n-1}, \quad v_{-2} \Rightarrow v_{n-2} \]

For open contours sufficient number of points must be fixed
Energy minimization through PDEs

- Extended model
- Idea: better physical model by taking temporal development of snake into account (it moves!)
- Add kinetic energy to total energy function

$$E_K(v) = \frac{1}{2} \int_0^1 \mu(s) \left( \frac{\partial v(s, t)}{\partial t} \right)^2 ds$$

- mass of snake
- temporal derivative

Total energy

$$E(v) = E_K(v) + E_I(v) + E_D(v)$$
Further improvement: snake dissipates energy through friction

\[ D(v_t) = \frac{1}{2} \int_0^1 \gamma(s) |v_t|^2 \, ds \]

damping coefficient

- \( t \) now has the physical meaning of *time*, as snake moves in image
- Find where snake moves, subject to all forces, by min total energy

\[ \int_0^1 (E(v) + D(v_t)) \, ds \]

\[ \mu(s) \frac{\partial^2 v}{\partial t^2} + \gamma(s) \frac{\partial v}{\partial t} - \frac{\partial}{\partial s} \left( \alpha(s) \frac{\partial v}{\partial s} \right) + \frac{\partial^2}{\partial s^2} \left( \beta(s) \frac{\partial^2 v}{\partial s^2} \right) = -\nabla P(v) \]
Computer Vision

Spatio-temporal model

with weights uniform over the snake

\[
\mu \frac{\partial^2 x}{\partial t^2} + \gamma \frac{\partial x}{\partial t} - \alpha \frac{\partial^2 x}{\partial s^2} + \beta \frac{\partial^4 x}{\partial s^4} = - \frac{\partial P}{\partial x}
\]

\[
\mu \frac{\partial^2 y}{\partial t^2} + \gamma \frac{\partial y}{\partial t} - \alpha \frac{\partial^2 y}{\partial s^2} + \beta \frac{\partial^4 y}{\partial s^4} = - \frac{\partial P}{\partial y}
\]

discretize to points, and finite difference derivatives (also for time)

\[
v_t \approx v^t - v^{t-1}
\]

\[
v_{tt} \approx v^t - 2v^{t-1}v^{t-2}
\]

Get again a linear system of equations

\[
(\mu + \gamma)v^t_i - \alpha(v^t_{i-1} - 2v^t_i + v^t_{i+1}) + \beta(v^t_{i-2} - 4v^t_{i-1} + 6v^t_i - 4v^t_{i+1} + v^t_{i+2}) = - P_v \bigg|_{v^t_{i-1}} + (2\mu + \gamma)v^{t-1}_i - \mu v^{t-2}_i
\]
Spatio-temporal model

In matrix form

\[
[(\mu + \gamma)I + K]v^t = -P_v \bigg|_{v^{t-1}} + (2\mu + \gamma)v^{t-1} - \mu v^{t-2}
\]

Role of damping $\rightarrow$ better conditioned (extended) $K$

Solve by inverting (extended) $K$

\[
v^t = [(\mu + \gamma)I + K]^{-1} \left( -P_v \bigg|_{v^{t-1}} + (2\mu + \gamma)v^{t-1} - \mu v^{t-2} \right)
\]
Extension with further external forces

- Pressure (Baloons)
  - Pushing the curve to extend (inside to outside)
  - Constant force magnitude
  - Shows in the direction of the curve’s normal
- Gravity
  constant force in a preferred direction
Interactive forces
Snake example
Interactive forces example
Limitations

- May over-smooth the boundary

- Cannot follow topological changes of objects
Contour closing

Where are the gaps?
Limitations

- External energy: snake does not really “see” object boundaries in the image unless it gets very close to it.

image gradients $\nabla I$
are large only very near the boundary
• External energy can also be defined based on the **distance** from edges in the image.

Value at (x,y) tells how far that position is from the nearest edge point (or other binary image structure).
Snakes: Pros and Cons

Pros:
- Useful to track and fit non-rigid shapes
- Contour remains connected
- Possible to fill in “subjective” contours
- Flexibility in how energy function is defined, weighted

Cons:
- Must have good initialization near true boundary, may get stuck in undesired local minimum
- Parameters of energy function must be set well based on prior information
Summary: main points

- Deformable shapes and active contours are useful for
  - Segmentation: fit or “settle” to boundary in image
  - Tracking: previous frame’s estimate serves to initialize the next

- Optimization for snakes: general idea of minimizing an energy function
  - Can define terms to encourage certain shapes, smoothness, low curvature, push/pulls, …
  - Can use weights to control relative influence of each component

- Edges / optima in gradients can act as “attraction” force for interactive segmentation methods.

- Distance transform definition: efficient map of distances to nearest feature of interest.